Quantum Inequalities That Test Locality

Dennis Dieks
Institute for the History and Foundations of Science
Utrecht University, P.O.Box 80.000
3508 TA Utrecht, The Netherlands
Email: d.g.b.j.dieks@phys.uu.nl

20 June 2002

Abstract

Quantum theory violates Bell's inequality, but not to the maximum extent that is logically possible. We derive inequalities (generalizations of Cirel'son's inequality) that quantify this observation, both for the standard formalism and the formalism of generalized observables (POVMs). These are quantum analogues of Bell inequalities, and we show that they can be used to test quantum mechanical locality. We discuss the nature of this kind of locality. We also go into the relation of our results to a well-known argument by Popescu and Rohrlich [3] in which a fundamental connection between the existence of Cirel'son's bound and locality is denied.

1 Introduction

The violation of Bell's inequality by the predictions of quantum theory (both in its non-relativistic and relativistic versions) shows that quantum theory is non-local in the sense that its results cannot be reproduced by a local hidden-variables theory. However, the maximum violation of Bell's inequality that is allowed by quantum theory is less than the maximum violation that is logically possible—quantum theory obeys Cirel'son's inequality. One might surmise that this fact, namely that quantum theory does not violate Bell's inequality more than it actually does, is due to the fact that quantum theory does not abandon locality completely: after all, quantum theory does nor permit signals that go faster than light. Perhaps compliance with relativistic causality restricts the extent to which Bell's inequality can be violated, and perhaps inequalities like Cirel'son's can be regarded as a touchstone of this kind of locality (in the same way as Bell's inequality is a touchstone for locality in the classical sense).

In this paper we argue that this hypothesis is true: the fact that quantum theory does not violate Bell's inequality to the maximum extent logically possible is due to the features of locality that are built into the theory. We derive a set of quantum Bell inequalities, and a strongest inequality representing this whole set, that can be regarded as quantum versions of Bell's inequality. To make the analogy with Bell inequalities clear we will analyze how locality is implemented in quantum theory, and how the inequalities we derive are based on locality assumptions. We will discuss how these results relate to an argument of Popescu and Rohrlich ([3])that might seem to show that the existence of Cirel'son's bound is unconnected with locality issues.

2 Cirel'son's inequality

Consider a probability space in which there are four stochastic functions, A, a, B, b, each of which can take the values $+1$ or -1. The quantity $AB + Ab + aB - ab = A(B + b) + a(B - b)$ can only be $+2$ or -2, from which it follows that the absolute value of its expectation value is smaller than 2:

$$|< AB + Ab + aB - ab >| \leq 2 \ . \quad (1)$$

This is the form of Bell's inequality that we will consider. The inequality is respected by classical physical theories, in which the physical magnitudes are represented by variables that possess a joint probability distribution—we will discuss the connection with locality in sect. 5.

In quantum mechanics physical magnitudes are not represented by stochastic functions on a phase space, but by Hermitean operators on a Hilbert space. Let us in this section use letters like A, a, B, b to denote such operators that have eigenvalues $+1$ and -1, and let us consider a combination of them that is analogous to the combination of quantities in Bell's inequality: $AB + Ab + aB - ab$, where we assume that the operators occurring in a product commute. As was first shown by Cirel'son [1], the modulus of the quantum mechanical expectation value of this expression is bounded by $2\sqrt{2}$: $| < AB + Ab + aB - ab > | \leq 2\sqrt{2}$—the upper bound can be attained, for example in the singlet state. So Bell's inequality can be violated by quantum theory; but the quantum expectation value stays well below the logically possible upper bound of the expression $| < AB + Ab + aB - ab > |$, which is 4.

Cirel'son's inequality can be proved elegantly by observing ([2]) that if $A^2 = a^2 = B^2 = b^2 = \mathbb{1}$ and $[A, B] = [A, b] = [a, B] = [a, b] = 0$, then $C^2 \equiv (AB + Ab + aB - ab)^2 = 4\mathbb{1} - [A, a].[B, b]$. It follows from this that $< C >^2 \leq < C^2 > \leq ||C||^2 \leq 4 + 4||A||.||a||.||B||.||b|| = 8$, or $| < C > | \leq 2\sqrt{2}$.

An alternative simple proof, which is analogous to the above proof of Bell's inequality (1) and similar to proofs of other inequalities that we shall discuss in sections 3 and 4), goes as follows.

Put $A|\psi\rangle \equiv |A\rangle$, $B|\psi\rangle \equiv |B\rangle$, $a|\psi\rangle \equiv |a\rangle$ and $b|\psi\rangle \equiv |b\rangle$. Each of these four vectors has a norm that is ≤ 1. We now have

$$\begin{aligned} | < C > | &= |\langle\psi|C|\psi\rangle| \\ &= |\langle A|B + b\rangle + \langle a|B - b\rangle| \leq || \, |B\rangle + |b\rangle \, || + || \, |B\rangle - |b\rangle \, || \\ &\leq \sqrt{2(1 + \mathbf{Re}\langle B|b\rangle)} + \sqrt{2(1 - \mathbf{Re}\langle B|b\rangle)} \leq 2\sqrt{2} \qquad (2) \end{aligned}$$

.

The important difference between this derivation and the derivation of Bell's inequality is that for *numbers* B and b with norm ≤ 1 we have $|(B + b) + (B - b)| \leq 2$, whereas for *vectors* with norm ≤ 1 we find $||(|B\rangle + |b\rangle) + (|B\rangle - |b\rangle)|| \leq 2\sqrt{2}$. In the latter case the maximum is attained when $|B\rangle$ and $|b\rangle$ are perpendicular.

3 Generalized measurements

Above we followed the orthodox point of view concerning the mathematical representation of physical quantities in quantum theory: physical quantities are represented by hermitian operators. Within this framework joint measurability is equivalent to commutativity (which in turn leads to causality in the context of the EPR experiment). But there is a more general treatment of measurements in quantum theory, first developed by Ludwig [4] and Davies [5], in which physical quantities correspond to collections of positive operators M_i on the Hilbert space, such that

$$M_i \geq 0, \ \ \sum_i M_i = \mathbb{1}.$$

If the possible outcomes of a measurement of the considered quantity are m_i, the probabilities of obtaining these values in a state ρ of the system are given by $\mathrm{Tr}\rho M_i$. The mapping $m_i \rightarrow M_i$ is a 'positive operator valued mapping' (POVM) representative of the associated physical quantity $\mathcal{M}$.

Two physical quantities $\mathcal{A}$ and $\mathcal{B}$, represented by sets of positive operators $\{A_i\}$ and $\{B_j\}$, respectively, are jointly measurable if there is a third quantity $\mathcal{O}$, represented by $\{O_k\}$, such that

$$A_i = \sum_{k \in K_i} O_k, \quad B_j = \sum_{k \in K'_j} O_k, \tag{3}$$

where $\{K_i\}$ and $\{K'_j\}$ are two partitions of the index set through which k runs.

If there is an $\mathcal{O}$ satisfying Eq.(3) we can measure it, and infer information about the outcomes and their probabilities of both $\mathcal{A}$ and $\mathcal{B}$ by grouping together the results according to the two partitions. An important result is now that commutativity of the two generalized observables $\mathcal{A}$ and $\mathcal{B}$ (in the sense that $A_i B_j = B_j A_i$ for all i, j) is a sufficient but not a necessary condition for their joint measurability. *If* $\mathcal{A}$ and $\mathcal{B}$ commute, the products $A_i B_j$ are positive operators characterizing a joint measurement. But in general a joint measurement need not correspond to product operators.

So in the EPR situation we could imagine a joint measurement of two non-commuting generalized observables $\mathcal{A}$ and $\mathcal{B}$, each pertaining to a different wing of the experiment. In this case it would no longer be true that the mere requirement of compatibility leads to causal independence, the product form of the joint measurement operators, and the validity of Cirel'son's inequality. Indeed, in general it will be possible to find four generalized joint observables (for the four possible combinations of quantities at the individual wings) that give rise to maximum violation of the inequality.

However, Busch has shown [7] that if causal independence is required, i.e. that the measurement results and probabilities represented by a generalized observable at one wing are independent of which generalized observable is measured at the other side, the generalized observables at one wing must commute with those at the other, and the observable corresponding to the joint measurement takes on the product form again. So within the generalized measurements framework commutativity and product form are direct expressions of causality in the sense of impossibility of signaling.

If the measurements in the EPR experiment are represented by generalized observables, and if locality in the form of impossibility of signaling is assumed, Cirel'son's inequality can again be derived. To see this, consider one pair of the four pairs of observables: $\mathcal{A}$ and $\mathcal{B}$. Because of the no-signaling requirement, the corresponding positive operators A_i and B_j commute, and the joint measurement of $\mathcal{A}$ and $\mathcal{B}$ can be represented by four positive operators $A_i B_j$, with $i, j = \pm 1$. The expectation value of the outcomes of this joint measurement, in the pure state $|\psi\rangle$, becomes:

$$\langle\psi|(A_1 B_1 - A_1 B_{-1} + A_{-1} B_{-1} - A_{-1} B_1)|\psi\rangle = \tag{4}$$

$$\langle\psi|(A_1 - A_{-1})(B_1 - B_{-1})|\psi\rangle = \langle A|B\rangle, \tag{5}$$

where $|A\rangle \equiv (A_1 - A_{-1})|\psi\rangle$ and $|B\rangle \equiv (B_1 - B_{-1})|\psi\rangle$. So for the purpose of calculating expectation values the generalized observables $\mathcal{A}, \mathcal{B}, \tilde{a}, \tilde{b}$ can each be represented by a single hermitian operator, namely $(A_1 - A_{-1})$, $(B_1 - B_{-1})$, $(a_1 - a_{-1})$ and $(b_1 - b_{-1})$, respectively; the joint measurements are represented by the corresponding products. Compared to the case discussed in sect. 2, the only differences are that the operators $A_i, B_j, \ldots$ need not be projection operators, and the squares of $(A_1 - A_{-1}), (B_1 - B_{-1}), \ldots$ need not be $\mathbb{1}$. The second proof of the Cirel'son inequality given in section 2 goes nevertheless through, because the operators $(A_1 - A_{-1}), (B_1 - B_{-1}), \ldots$ have a norm ≤ 1.

Indeed,

$$||(A_1 - A_{-1})|\psi\rangle||^2 = ||A_1|\psi\rangle||^2 + ||A_2|\psi\rangle||^2 - 2\langle A_1\psi|A_2\psi\rangle, \tag{6}$$

whereas

$$||(A_1 + A_{-1})|\psi\rangle||^2 = ||A_1|\psi\rangle||^2 + ||A_2|\psi\rangle||^2 + 2\langle A_1\psi|A_2\psi\rangle = 1, \tag{7}$$

so that

$$||(A_1 - A_{-1})|\psi\rangle||^2 = 1 - 4\langle A_1\psi|A_2\psi\rangle. \qquad (8)$$

Because $A_2 = \mathbb{1} - A_1$, $[A_1, A_2] = 0$ and the inner products in (6), (7) and (8) are real. This inner product is also ≥ 0:

$$\langle A_1\psi|A_2\psi\rangle = \langle\psi|A_1|\psi\rangle - \langle\psi|A_1^2|\psi\rangle \qquad (9)$$

which is ≥ 0 because A_1 has norm ≤ 1 and therefore possesses eigenvalues λ_i with $0 \leq \lambda_i \leq 1$.

Now introduce vectors $|a\rangle, |b\rangle$ in the obvious way: $|a\rangle \equiv (a_1 - a_{-1})|\psi\rangle, |b\rangle = (b_1 - b_{-1})|\psi\rangle$; as just shown, the vectors $|A\rangle, |B\rangle, |a\rangle, |b\rangle$ all have norms ≤ 1, as in sect. 2. We therefore find:

$$| < \mathcal{AB} + \mathcal{A}\tilde{b} + \tilde{a}\mathcal{B} - \tilde{a}\tilde{b} > | = |\langle A|B + b\rangle + \langle a|B - b\rangle| \leq 2\sqrt{2}. \qquad (10)$$

The inequality therefore holds in every pure state $|\psi\rangle$. Its validity in any mixed state ρ follows immediately.

4 The strongest inequality

Cirel'son's inequality is not the only and not the strongest inequality that can be derived from the requirement of commutativity. Put $X \equiv < \mathcal{A}\tilde{b} + \tilde{a}\mathcal{B} >$ and $Y \equiv < \mathcal{AB} - \tilde{a}\tilde{b} >$. Now Cirel'son's inequality can be written as

$$|X + Y| \leq 2\sqrt{2}. \qquad (11)$$

So in the X, Y 'correlation plane' Cirel'son's inequality restricts the points (X, Y) to the strip between the two lines

$$X + Y = \pm 2\sqrt{2}. \qquad (12)$$

But by a minimal change in the proof of sect. 2 it follows that also the following inequality holds:

$$|X - Y| \leq 2\sqrt{2}, \qquad (13)$$

so that the points must also lie in the strip bounded by the lines

$$X - Y = \pm 2\sqrt{2}. \qquad (14)$$

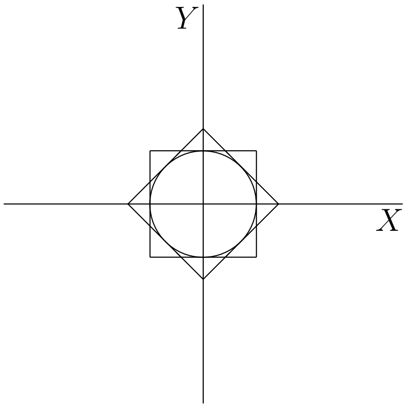

Figure 1: The X, Y plane. The slanted square represents inequalities (11) plus (13), the circle inequality (16).

We also have the obvious inequalities $|X| \leq 2, \quad |Y| \leq 2$, so that the allowed points (X, Y) must be in the intersection of the interiors of the two squares indicated in Fig. 1. It further turns out that they must be inside (or on the sides of) all squares that result from these just-mentioned squares by applying an arbitrary rotation around an axis through the origin of the X, Y plane and perpendicular to it. To prove this, consider the expression $|X \cos\varphi + Y \sin\varphi|$. We have:

$$\begin{aligned} |X \cos\varphi + Y \sin\varphi| &= |\langle A|B\cos\varphi + b\sin\varphi\rangle + \langle a|B\sin\varphi - b\cos\varphi\rangle| \\ &\leq ||\,|B\cos\varphi\rangle + |b\sin\varphi\rangle|| + ||\,|B\sin\varphi\rangle - |b\cos\varphi\rangle|| \\ &\leq \sqrt{\sin^2\varphi + \cos^2\varphi + 2\mathbf{Re}\langle B|b\rangle \sin\varphi\cos\varphi} \\ &\quad + \sqrt{\sin^2\varphi + \cos^2\varphi - 2\mathbf{Re}\langle B|b\rangle \sin\varphi\cos\varphi} \\ &\leq \sqrt{\sin^2\varphi + \cos^2\varphi} + \sqrt{\sin^2\varphi + \cos^2\varphi} = 2. \quad (15) \end{aligned}$$

Cirel'son's inequality and the other inequalities mentioned earlier in this section are special cases of this general set of inequalities (in which φ can take arbitrary values). It should be noted that our proofs apply both to the case of ordinary observables and the case of generalized observables.

It is clear from the geometry of Fig. 1 that the requirement that all the inequalities (15) be satisfied leads to the inequality

$$X^2 + Y^2 = < \mathcal{A}\tilde{b} + \tilde{a}\mathcal{B} >^2 + < \mathcal{A}\mathcal{B} - \tilde{a}\tilde{b} >^2 \leq 4. \quad (16)$$

All points X, Y are inside or on the circumference of a circle with radius 2.

Inequality (16) (which was recently proved directly, by a variational argument, for the case of ordinary spin observables by Uffink [8]) summarizes all generalized Cirel'son inequalities 15. All values X, Y that satisfy (16) also satisfy all Cirel'son inequalities (15); but satisfaction of a finite number of inequalities of (15) is not sufficient to guarantee satisfaction of (16). Moreover, each point on the circumference of the circle can actually be attained, because the bound of the corresponding generalized Cirel'son inequality can be attained (the one resulting in a line tangent to the circle in the point in question). Inequality (16) is therefore the strongest one that follows from the requirement of local commutivity.

5 Locality

Bell's inequality (1) is valid for an arbitrary quadruple of stochastic functions on a probability space, and as such is not immediately connected with locality issues. The link with locality comes in via the application of (1) to situations of the Einstein-Podolsky-Rosen type: A and a, and B and b, stand for measurements on the space-like separated wings 1 and 2, respectively. An experimenter at 1 can choose between measuring A and a; her or his colleague at 2 has the choice between B and b. The combined measurement on parts 1 and 2 is represented by the product of the individual result functions, on the basis of a locality assumption: a measurement of a physical quantity on one wing of the experiment can be represented by one and the same function, regardless of which measurement is performed at the other side. Both the possible measurement outcomes and their probabilities are insensitive to which choice of physical quantity is made at the other side.

An obvious additional assumption is that A, a, B, b correspond to characteristic measurement devices and procedures. The device corresponding to A, e.g. should remain the same in different instances—that the possible outcomes and corresponding probabilities remain the same is by itself not enough. Consider, for example, a Stern-Gerlach device that undergoes a rotation depending on the choice between B and b: although the possible outcomes would still be $+1$ and -1 and the probabilities would remain equal to $1/2$, if the device measures the spin of a spin-1/2 particle, this would not constitute one specific measured quantity. Spin along different axes would be measured, and in spite of the fact that the same function A could be written down, the rotation of the corresponding device would signal non-locality. The same applies to quantum theory: in any expression in which hermitian operators A, B, ..., occur, these refer to definite physical quantities linked to characteristic measuring procedures.

The features of locality mentioned above in the context of classical theory have exact counterparts in the quantum treatment. The same operators are used to represent physical quantities on the wings of the experiment, regardless of what goes on at the other side. This makes sense because the operators at the two wings are assumed to be compatible: they commute, so that their product is again an observable. Further, because the operators commute, the probabilities of outcomes on the two sides of the experiment are insensitive to what happens at the other wing. This is the (relativistic) causality mentioned in the Introduction. These features of locality are sufficient to derive Cirel'son's inequality and its generalizations, including inequality (16). Indeed, the only premise needed to derive these inequalities was the possibility to represent the joint measurements at the wings by the products of commuting one-wing operators. Both within the framework of ordinary and generalized observables this is equivalent to the requirement of causal independence (in the sense of the impossibility of signaling). Violation of inequality (16) or of the weaker other inequalities would therefore imply an influence of the measurements at the two wings on each other; the inequality can be used for experimental tests of causality.

This statement might seem in conflict with an argument by Popescu and Rohrlich ([3]). These authors argue that relativistic causality does *not* limit the sum of the correlations occurring in Cirel'son's inequality to $2\sqrt{2}$. They do so by considering an EPR situation, in which spin measurements are performed on the two wings. The two possible outcomes are $+1$ and -1 along any axis, and both possibilities have probability $1/2$. For any pair of axes, the combinations of outcomes $+1, +1$ and $-1, -1$ are equally probable, and the same applies to the combinations $+1, -1$ and $-1, +1$. So the outcomes and probabilities on one wing of the experiment are independent of the measurements done at the other side: relativistic causality is satisfied. Now suppose that the correlation function (a 'superquantum' correlation function) has a form like the following:

$$E(\theta) = \begin{cases} +1 & \text{for } 0 \leq \theta \leq \pi/4 \\ 2 - 4\theta/\pi & \text{for } \pi/4 \leq \theta \leq 3\pi/4 \\ -1 & \text{for } 3\pi/4 \leq \theta \leq \pi \end{cases} \tag{17}$$

This is equivalent to assuming that the probability $p_{++}(\theta)$ of the pair of outcomes $+1, -1$ is given by

$$p_{++}(\theta) = \frac{E(\theta)+1}{4}.$$

In these formulas θ is the angle between the axes on the left and right, respectively, along which the spin measurements are made.

Now consider four axes $\alpha', \beta, \alpha, \beta'$ separated by successive angles of $\pi/4$ and lying in one plane. We find that

$$E(\alpha,\beta) + E(\alpha',\beta) + E(\alpha,\beta') - E(\alpha',\beta') = 4. \quad (18)$$

Apparently, Cirel'son's inequality can be violated, even to the maximum extent logically possible, by a correlation function that respects relativistic causality.

However, the above correlation function is not the one predicted by quantum theory. Indeed, we have seen that within the framework of theories that operate with a phase space on which physical quantities are represented by (stochastic) functions, or within the Hilbert space formalism of quantum theory, a correlation function that violates Cirel'son's inequality cannot arise in a local way. The only way to realize the violation of Eq.(18) within quantum mechanics would be to *change*, depending on the angles chosen, the physical quantities that enter into the definition of $E(\theta)$. A trivial example can be constructed by stipulating that the concrete physical implementation of measuring $E(\alpha,\beta)$ in the two-particle singlet state be "measure σ_x^I and $-\sigma_x^{II}$, call the results 'spin of particle I along α and spin of particle II along β', respectively, and determine the expectation value of the product", similarly for α', β and α, β', and inserting a minus sign for α', β'. Obviously, the correlation function defined in this way satisfies all mathematical requirements imposed by Popescu and Rohrlich (including that of relativistic causality), and violates Cirel'son's inequality maximally. But within the framework of quantum mechanics it does not represent a correlation between local quantities in the sense we have discussed.

The fact that within the framework of quantum theory inequality (16) and the generalized Cirel'son inequalities reflect locality does not mean, of course, that there cannot exist different theoretical frameworks in which the above 'superquantum' correlation function *would* arise in a local way; frameworks that use neither functions on a state space nor the Hilbert space operator formalism. Popescu's and Rohrlich's argument does show (and that is all they claimed) that the inequalities do not follow from the requirement of relativistic causality *alone*. Further assumptions are needed about the way in which physical quantities are represented. Locality and causality as fleshed out within classical theories lead to Bell's inequality; as fleshed out in quantum theory they lead to inequality (16).

6 Conclusion

References

[1] B.S. Cirel'son, *Lett. Math. Phys.* **4** (1980) 93.

[2] L.J. Landau, *Phys. Lett.* **A 120** (1987) 52.

[3] S. Popesu and D. Rohrlich, *Found. Phys.* **24** (1994) 379.

[4] G. Ludwig, *Einführung in die Grundlagen der Theoretischen Physik*, Vol. 3 (Vieweg, Braunschweig, 1976).

[5] E.B. Davies, *Quantum Theory of Open Systems* (Academic Press, New York, 1976).

[6] J. Uffink, *Int. Jnl. Theor. Phys.* **33** (1994) 199.

[7] P. Busch and J. Singh, *Phys. Lett. A* **249** (1998)10.

[8] J. Uffink, *Phys. Rev. Lett.* 88 (2002) 230406.

[9] A. Cabello, *Phys. Rev. Lett.* **88** (2002) 060403.

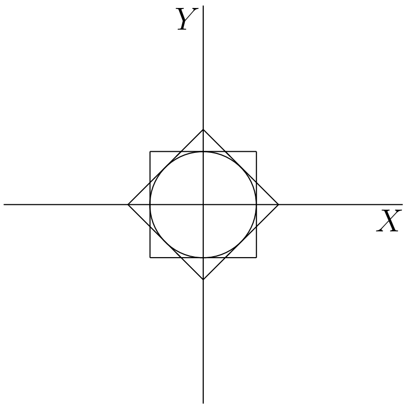

Figure 2: The X, Y plane. The slanted square represents inequalities (11) plus (13), the circle inequality (16).