Classification and monogamy of three-qubit biseparable Bell correlations

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We strengthen the set of Bell-type inequalities presented by Sun and Fei [Phys. Rev. A **74**, 032335 (2006)] that give a classification for biseparable correlations and entanglement in tripartite quantum systems. We will furthermore consider the restriction to local orthogonal spin observables and show that this strengthens all previously known such tripartite inequalities. The quadratic inequalities we find indicate a type of monogamy of maximal biseparable tripartite quantum correlations, although the nonmaximal ones can be shared. This is contrasted to recently found monogamy inequalities for bipartite Bell correlations in tripartite systems.

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I. INTRODUCTION

Although Bell inequalities have been originally proposed to test quantum mechanics against local realism, nowadays they also serve another purpose, namely, investigating quantum entanglement. Indeed, Bell inequalities were used to give detailed characterizations of multipartite entangled states by giving bounds on the correlations that these states can give rise to [1].

Recently a set of Bell-type inequalities was presented by Sun and Fei [2] that gives a finer classification for entanglement in tripartite systems than was previously known. The inequalities distinghuish three different types of bipartite entanglement that may exist in tripartite systems. They not only determine if one of the three parties is separable with respect to the other two, but also which one. It was shown that the three inequalities give a bound that can be thought of as tracing out a sphere in the space of expectations of the three Bell operators that were used in the inequalities. Here we strenghten this bound by showing that all states are confined within the interior of the intersection of three cylinders and the already mentioned sphere.

Furthermore, in Refs. [3,4] it was shown that considerably stronger separability inequalities for the expectation of Bell operators can be obtained if one restricts oneself to local orthogonal spin observables (so-called LOOs [5]). We will show that the same is the case for the Bell operators considered here by strengthening all above-mentioned tripartite inequalities under the restriction of using orthogonal observables.

The relevant tripartite inequalities are included in the N-particle inequalities derived in [6]. It was shown that these N-partite inequalities can be violated maximally by the N-particle maximally entangled Greenberger-Horne-Zeilinger (GHZ) states [6], but, as will be shown here, they can also be maximally violated by states that contain only $(N-1)$-partite entanglement. Although these inequalities thus allow for further classification of multipartite entanglement (besides some other interesting properties), they cannot be used to distinghuish full N-particle entanglement from $(N-1)$-particle entanglement in N-partite states. It is shown that this is not the case for the stronger bounds that are derived for the case of LOOs.

In Sec. II the analysis for unrestricted spin observables is performed and in Sec. III for the restriction to LOOs. Lastly, in the discussion of Sec. IV we will interpret the presented quadratic inequalities as indicating a type of monogamy of maximal biseparable three-particle quantum correlations. Nonmaximal correlations can, however, be shared. This is contrasted to the recently found monogamy inequalities of Toner and Verstraete [7].

II. UNRESTRICTED OBSERVABLES

Chen *et al.* [6] consider N-parties that each have two alternative dichotomic measurements denoted by A_j and A_j' (outcomes ±1) and show that local realism (LR) requires that

$$|\langle D_{\rm LR}^{(i)}\rangle| = \frac{1}{2}|\langle B_{N-1}^{(i)}(A_i + A_i') + (A_i - A_i')\rangle_{\rm LR}| \leq 1, \quad (1)$$

for $i=1,2,\ldots,N$, where $B_{N-1}^{(i)}$ is the Bell polynomial of the Werner-Wolf-Zukowski-Brukner (WWZB) inequalities [8] for the $N-1$ parties, except for party i. These Bell-type inequalities have only two different local settings and are contained in the general inequalities for $N>2$ parties that have more than two alternative measurement settings derived by Laskowski *et al.* [9]. Indeed, they follow from the latter when choosing certain settings equal. Note, furthermore, that the WWZB inequalities are contained in the inequalities of Eq. (1) by choosing $A_N = A_N'$.

The quantum mechanical counterpart of the Bell-type inequality of Eq. (1) is obtained by introducing dichotomic observables A_k, A_k' for each party k. Let us define analogously to Sun and Fei [2] the operator

$$\mathcal{D}_N^{(i)} = \mathcal{B}_{N-1}^{(i)} \otimes (A_i + A_i')/2 + \mathbb{1}_{N-1}^{(i)} \otimes (A_i - A_i')/2, \quad (2)$$

for $i=1,2,\ldots,N$. Here $\mathcal{B}_{N-1}^{(i)}$ and $\mathbb{1}_{N-1}^{(i)}$ are, respectively, the Bell operator of the WWZB inequalities and the identity operator both for the $N-1$ qubits not involving qubit i.

The quantum mechanical counterpart of the local realism inequalities of Eq. (1) for all i is then

$$|\langle \mathcal{D}_N^{(i)}\rangle| \leq 1, \quad (3)$$

where $\langle \mathcal{D}_N^{(i)}\rangle := {\rm Tr}[\mathcal{D}_N^{(i)}\rho]$ and ρ is a N-party quantum state.

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Since the Bell inequality of Eq. (3) uses only two alternative dichotomic observables for each party, the maximum violation of this Bell inequality is obtained for an N-particle pure qubit state and furthermore for projective observables, as proven recently by Masanes [10] and by Toner and Verstraete [7]. In the following we will thus consider qubits only and the observables will be represented by the spin operators $A_k = \boldsymbol{a}_k \cdot \boldsymbol{\sigma}$ and $A'_k = \boldsymbol{a}'_k \cdot \boldsymbol{\sigma}$ with $\boldsymbol{a}_k$ and $\boldsymbol{a}'_k$ unit vectors that denote the measurement settings and $\boldsymbol{a} \cdot \boldsymbol{\sigma} = \sum_l a_l \sigma_l$, where σ_l are the familiar Pauli spin observables for $l = x$, y, and z on $\mathcal{H} = \mathbb{C}^2$. In fact, it suffices [7] to consider only real and traceless observables, so we can set $a_y = 0$ for all observables.

An interesting feature of the inequalities in Eq. (3) is that all generalized GHZ states $|\psi_\alpha^N\rangle = \cos \alpha |0\rangle^{\otimes N} + \sin \alpha |1\rangle^{\otimes N}$ can be made to violate them for all α [6,9], which is not the case for the WWZB inequalities. Furthermore, the maximum is given by

$$\max_{A_i, A'_i} |\langle \mathcal{D}_N^{(i)} \rangle| = 2^{(N-2)/2}, \tag{4}$$

as was proven by Chen *et al.* [6]. They also noted that this maximum is obtained for the maximally entangled N-particle GHZ state $|\mathrm{GHZ}_N\rangle$ (i.e., $\alpha = \pi/4$) and for all local unitary transformations of this state. However, not noted in [6] is the fact that the maximum is also obtainable by N-partite states that only have $(N-1)$-particle entanglement, which is the content of the following theorem.

Theorem 1. Not only can the maximum value of $2^{(N-2)/2}$ for $\langle \mathcal{D}_N^{(i)} \rangle$ be reached by fully N-particle entangled states (proven in [6]) but also by N-partite states that only have $(N-1)$-particle entanglement.

Proof. First, $(\mathcal{B}_{N-1}^{(i)})^2 \leqslant 2^{(N-2)} \mathbb{1}_{N-1}^{(i)}$ (as proven in [8]). Here $X \leqslant Y$ means that $Y - X$ is semipositive definite. Thus the maximum possible eigenvalue of $\mathcal{B}_{N-1}^{(i)}$ is $2^{(N-2)/2}$. Consider a state $|\Psi_{N-1}^{(i)}\rangle$ for which $\langle \mathcal{B}_{N-1}^{(i)} \rangle_{|\Psi_{N-1}^{(i)}\rangle} = 2^{(N-2)/2}$. This must be [8] a maximally entangled $(N-1)$-particle state (for the N particles except for particle i), such as the state $|\mathrm{GHZ}_{N-1}\rangle$. Next consider the state $|\xi^{(i)}\rangle = |\Psi_{N-1}^{(i)}\rangle \otimes |0_i\rangle$, with $|0_i\rangle$ an eigenstate of the observable A_i with eigenvalue 1. This is an N-partite state that only has $(N-1)$-particle entanglement. Furthermore, choose $A_i = A'_i$ in Eq. (2). We then obtain $\langle \mathcal{D}_N^{(i)} \rangle_{|\xi^{(i)}\rangle} = \langle \mathcal{B}_{N-1}^{(i)} \rangle_{|\Psi_{N-1}^{(i)}\rangle} \langle A_i \rangle_{|0_i\rangle} = 2^{(N-2)/2}$, which was to be proven. ■

This theorem thus shows that the Bell inequalities of Eq. (3) cannot distinghuish between full N-partite entanglement and $(N-1)$-partite entanglement, and thus cannot serve as full N-particle entanglement witnesses.

Let us now concentrate on the tripartite case ($N=3$ and $i=1,2,3$). Sun and Fei [2] obtain that for fully separable three particle states it follows that $|\langle \mathcal{D}_3^{(i)} \rangle| \leqslant 1$, which does not violate the local realistic bound of Eq. (1). General three particle states give $|\langle \mathcal{D}_3^{(i)} \rangle| \leqslant \sqrt{2}$, which follows from Eq. (4). As follows from Theorem 1 this can be saturated by both fully entangled three particle states as well as for biseparable entangled three particle states (e.g., two-partite entangled three particle states).

Sun and Fei have furthermore presented a set of Bell inequalities that distinguish three possible forms of biseparable entanglement. They consider biseparable states that allow for the partitions 1–23, 2–13, and 3–12 where the set of states in these partitions is denoted as $S_{1-23}, S_{2-13}, S_{3-12}$ and which we label by $j=1,2,3$, respectively. These sets contain states such as $\rho_1 \otimes \rho_{23}, \rho_2 \otimes \rho_{13}$, and $\rho_3 \otimes \rho_{12}$, respectively. For states in partition j (and for $i=1,2,3$) Sun and Fei obtained

$$|\langle \mathcal{D}_3^{(i)} \rangle| \leqslant \chi_{i,j}, \tag{5}$$

with $\chi_{i,j} = \sqrt{2}$ for $i=j$ and $\chi_{i,j} = 1$ otherwise.

They furthermore proved that for all three qubit states

$$\langle \mathcal{D}_3^{(1)} \rangle^2 + \langle \mathcal{D}_3^{(2)} \rangle^2 + \langle \mathcal{D}_3^{(3)} \rangle^2 \leqslant 3, \ \forall \rho. \tag{6}$$

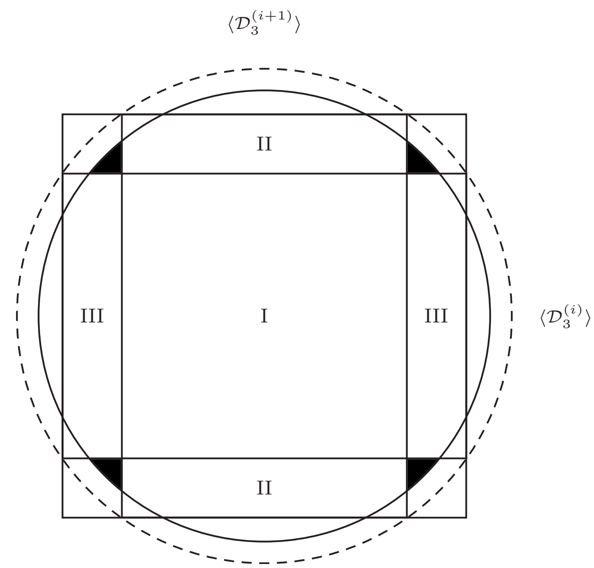

FIG. 1. $\mathcal{D}_3^{(i)} - \mathcal{D}_3^{(i+1)}$ plane with the stronger bound given by the circle with radius $\sqrt{5/2}$ which strengthens the less strong bound with radius $\sqrt{3}$ that is given by the dashed circle.

Although this inequality is stronger than the set above (for details see Fig. 1 in [2]), it can be saturated by fully separable states. For example, choose the state $|000\rangle$ and choose all observables to be projections onto this state. Then we get $\langle \mathcal{D}_3^{(1)} \rangle_{|000\rangle}^2 + \langle \mathcal{D}_3^{(2)} \rangle_{|000\rangle}^2 + \langle \mathcal{D}_3^{(3)} \rangle_{|000\rangle}^2 = 3$.

Let us consider $\mathcal{D}_3^{(i)}$ (for $i=1,2,3$) to be three coordinates of a space in the same spirit as Sun and Fei [2] did. They showed that the fully separable states are confined to a cube with edge length 2 and the biseparable states in partition $j=1,2,3$ are confined to cuboids with size either $2\sqrt{2} \times 2 \times 2, 2 \times 2\sqrt{2} \times 2$, or $2 \times 2 \times 2\sqrt{2}$. Note that states exist that are biseparable with respect to all three partitions (and thus must lie within the cube with edge length 2), but which are not fully separable [11]. Furthermore, all three-qubit states are in the intersection of the cube with size $2\sqrt{2}$ and of the sphere with radius $\sqrt{3}$. Sun and Fei note that this sphere is just the external sphere of the cube with edge 2, which is consistent with the above observation that fully separable states can lie on this sphere. If we look at the $\mathcal{D}_3^{(i)} - \mathcal{D}_3^{(i+1)}$ plane we get Fig. 1. The fully separable states are in region I; region II belongs to the biseparable states of partition $j=i+1$; and region III belongs to states of partition $j=i$. Other biseparable states and fully entangled states are outside these regions but within the circle with radius $\sqrt{3}$. However, in the following theorem we show a quadratic inequality even stronger than Eq. (6) which thus strengthens the bound in Fig. 1 given by the circle of radius $\sqrt{3}$ and which forces the states just mentioned into the black regions.

Theorem 2. For the case where each observer chooses between two settings all three qubit states obey the following inequality:

$$\langle \mathcal{D}_3^{(i)} \rangle^2 + \langle \mathcal{D}_3^{(i+1)} \rangle^2 \leq \frac{5}{2}, \forall \rho, \tag{7}$$

for $i=1,2,3$ and where i and $i+1$ are both modulo 3.

Proof. The proof uses the exact same steps of the proof of Eq. (6) as performed by Sun and Fei (i.e., proof of Theorem 2 in [2]) and can be easily performed for the left-hand side of Eq. (7) that contains only two terms instead of the three terms on the right-hand side of Eq. (6). This results in only a minor change in calculations [14]. Case (3) in this proof then has the highest bound of 5/2, whereas the other three cases give a lower bound equal to 2. ∎

Note that in contrast to Eq. (6) the inequality of Eq. (7) cannot be saturated by separable states, since the latter have a maximum of 2 for the left-hand expression in Eq. (7).

If we again look at the space given by the coordinates $\mathcal{D}_3^{(i)}$ (for $i=1,2,3$), we have thus found that all states are, firstly, confined within the intersection of the three orthogonal cylinders $\langle \mathcal{D}_3^{(i)} \rangle^2 + \langle \mathcal{D}_3^{(i+1)} \rangle^2 \leq 5/2$ (with i and $i+1$ both modulo 3) each with radius $\sqrt{5/2}$ and, secondly, they must furthermore still lie within the cube of edge length $2\sqrt{2}$, and thirdly they must also lie within the sphere with radius $\sqrt{3}$. In Fig. 1 we see the strengthened bound of Eq. (7) as compared to the bound of Sun and Fei. However, we see from this figure that neither the intersection of the three cylinders, nor the sphere, nor the cube, give tight bounds.

The black areas in Fig. 1 are nonempty. For the case of Eq. (7) states thus exist that have both $|\langle D_3^{(i)} \rangle| > 1$ and $|\langle D_3^{(i+1)} \rangle| > 1$ (for some i). For example, the so-called W-state

$$|W\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3} \tag{8}$$

gives $|\langle D_3^{(i)} \rangle| = 1.022$ for all i when the observables are chosen as follows: $A_i = \cos \alpha_i \sigma_z + \sin \alpha_i \sigma_x$ with $\alpha_i = -0.133$ and $A_i' = \cos \beta_i \sigma_z + \sin \beta_i \sigma_x$ with $\beta_i = 0.460$.

III. RESTRICTION TO LOCAL ORTHOGONAL SPIN OBSERVABLES

In Refs. [3,4] it was shown that considerably stronger separability inequalities for the expectation of the bipartite Bell operator $\mathcal{B}_2$ can be obtained if one restricts oneself to local orthogonal observables (LOOs). We will now show that the same is the case for the Bell operator $\mathcal{D}_3^{(i)}$. The following theorem strenghtens all previous bounds of Sec. II for general observables.

Theorem 3. Suppose all local observables are orthogonal, i.e., $\boldsymbol{a}_i \cdot \boldsymbol{a}_i' = 0$, then the following inequalities hold:

(i) for all states: $|\langle \mathcal{D}_3^{(i)} \rangle| \leq \sqrt{3/2} \approx 1.225$;

(ii) for fully separable states: $|\langle \mathcal{D}_3^{(i)} \rangle| \leq \sqrt{3/4} \approx 0.866$;

(iii) for biseparable states in partition $j=1,2,3$:

$$|\langle \mathcal{D}_3^{(i)} \rangle| \leq \chi_{i,j}, \tag{9}$$

with $\chi_{i,j} = \sqrt{3/2} \approx 1.225$ for $i=j$ and $\chi_{i,j} = \sqrt{3/4} \approx 0.866$ otherwise; and

(iv) lastly, for all states:

$$\langle D_3^{(i)} \rangle^2 + \langle D_3^{(i+1)} \rangle^2 \leq 2. \tag{10}$$

Proof. (i) The square of $\mathcal{D}_3^{(i)}$ is given by

$$(\mathcal{D}_3^{(i)})^2 = (\mathcal{B}_2^{(i)})^2 \otimes \frac{1}{2}(1 + \boldsymbol{a}_i \cdot \boldsymbol{a}_i')\mathbb{1}_i + \mathbb{1}_2^{(i)} \otimes \frac{1}{2}(1 - \boldsymbol{a}_i \cdot \boldsymbol{a}_i')\mathbb{1}_i, \tag{11}$$

where $\mathbb{1}_2^{(i)}$ is the identity operator for the 2 qubits not including qubit i. For orthogonal observables we get $\boldsymbol{a}_i \cdot \boldsymbol{a}_i' = 0$, and $(\mathcal{B}_2^{(i)})^2 \leq 2\mathbb{1}_2^{(i)}$ (as proven in [3,4]). The maximum eigenvalue of $(\mathcal{D}_3^{(i)})^2$ is thus 3/2, which implies that $|\langle \mathcal{D}_3^{(i)} \rangle| \leq \sqrt{3/2}$.

(ii) For fully separable states we have from Eq. (2) that

$$\langle \mathcal{D}_3^{(i)} \rangle = \frac{1}{2}[\langle \mathcal{B}_2^{(i)} \rangle \langle (A_i + A_i') \rangle + \langle (A_i - A_i') \rangle]. \tag{12}$$

Furthermore, for the case of orthogonal observables $|\langle \mathcal{B}_2^{(i)} \rangle| \leq 1/\sqrt{2}$ [3,4]. Thus $|\langle \mathcal{D}_3^{(i)} \rangle| \leq |[\langle (A_i + A_i') \rangle / \sqrt{2} + \langle (A_i - A_i') \rangle]/2|$. Since the averages are linear in the state ρ the maximum is obtained for a pure state of qubit i. This state can be represented as $1/2(\mathbb{1} + \boldsymbol{o} \cdot \boldsymbol{\sigma})$, with $|\boldsymbol{o}| = 1$ and $\boldsymbol{o} \cdot \boldsymbol{\sigma} = \Sigma_k o_k \sigma_k$ ($k = x,y,z$). Take $C = (A_i + A_i')$, $D = (A_i - A_i')$, and $\boldsymbol{s} = \boldsymbol{a}_i + \boldsymbol{a}_i'$, $\boldsymbol{t} = \boldsymbol{a}_i - \boldsymbol{a}_i'$. We get $|\boldsymbol{s}| = |\boldsymbol{t}| = \sqrt{2}$. Choose now without losing generality [7] $\boldsymbol{s} = \sqrt{2}(\cos\theta, 0, \sin\theta)$ and $\boldsymbol{t} = \sqrt{2}(-\sin\theta, 0, \cos\theta)$. Then

$$|\langle \mathcal{D}_3^{(i)} \rangle| \leq |(\boldsymbol{s} \cdot \boldsymbol{o}/\sqrt{2} + \boldsymbol{t} \cdot \boldsymbol{o})/2| = \left| \frac{1}{2}[(o_z - \sqrt{2}o_x)\sin\theta + (o_x + \sqrt{2}o_z)\cos\theta] \right|.$$

Maximizing over θ [i.e., $\max_\theta(X\cos\theta + Y\sin\theta) = \sqrt{X^2 + Y^2}$] and using $o_x^2 + o_y^2 + o_z^2 = 1$ we finally get

$$|\langle \mathcal{D}_3^{(i)} \rangle| \leq |\sqrt{3/4(o_x^2 + o_z^2)}| \leq \sqrt{3/4}. \tag{13}$$

(iii) For biseparable states in partition $j=i$ we get the same as in Eq. (12), but now $|\langle \mathcal{B}_2^{(i)} \rangle| \leq \sqrt{2}$. Using the method of (ii) we get

$$|\langle \mathcal{D}_3^{(i)} \rangle| \leq |(\sqrt{2}\boldsymbol{s} \cdot \boldsymbol{o} + \boldsymbol{t} \cdot \boldsymbol{o})/2| \leq \sqrt{3/2}. \tag{14}$$

For biseparable states in partition $i+1$ and $i+2$ a somewhat more elaborate proof is needed. Let us set $i=1$ and $j=3$ for convenience (for the other partition $j=2$ we get the same result). The maximum is again obtained for pure states. Every pure state in partition $j=3$ can be written as $|\psi\rangle = |\psi\rangle_{12} \otimes |\psi\rangle_3$. Then

$$|\langle\mathcal{D}_3^{(i)}\rangle| = \left|\frac{1}{4}\langle(A_1+A_1')(A_2+A_2')\rangle_{|\psi_{12}\rangle}\langle A_3\rangle_{|\psi_3\rangle} + \frac{1}{4}\langle(A_1+A_1')(A_2 - A_2')\rangle_{|\psi_{12}\rangle}\langle A_3'\rangle_{|\psi_3\rangle} + \frac{1}{2}\langle(A_1-A_1')\otimes \mathbb{1}_2\rangle_{|\psi_{12}\rangle}\right|. \quad (15)$$

Using the technique in (ii) above it is found that the maximum over $|\psi_3\rangle$ gives

$$|\langle\mathcal{D}_3^{(i)}\rangle| \leq \left|\frac{\sqrt{2}}{4}[\langle(A_1+A_1')A_2\rangle^2_{|\psi_{12}\rangle} + \langle(A_1+A_1')A_2'\rangle^2_{|\psi_{12}\rangle}]^{1/2} + \frac{1}{2}\langle(A_1-A_1')\otimes \mathbb{1}_2\rangle_{|\psi_{12}\rangle}\right|. \quad (16)$$

Without losing generality we choose A_i, A_i' in the $x-z$ plane [7] and $|\psi\rangle_{12}=\cos\theta|01\rangle+\sin\theta|10\rangle$. We can use the symmetry to set $A_1=A_2=A$ and $A_1'=A_2'=A'$. This gives

$$|\langle\mathcal{D}_3^{(i)}\rangle| \leq \left|\frac{1}{2}(a_z-a_z')\cos(2\theta) + \frac{\sqrt{2}}{4}\{(a_z+a_z')^2 + [(a_x+a_x')^2\sin(2\theta)]^2\}^{1/2}\right|. \quad (17)$$

Since the observables A and A' must be orthogonal (i.e., $\boldsymbol{a}\cdot\boldsymbol{a}'=0$), this expression obtains its maximum for $a_x=a_x'=1/\sqrt{2}$ and $a_z=-a_z'=1/\sqrt{2}$. We finally get

$$|\langle\mathcal{D}_3^{(i)}\rangle| \leq \frac{\sqrt{2}}{2}\cos(2\theta) + \frac{1}{2}\sin(2\theta) \leq \sqrt{3/4}. \quad (18)$$

(iv) We use the exact same steps of the proof of Sun and Fei of Eq. (6) (i.e., proof of Theorem 2 in [2]) but since the observables are orthogonal only case (4) of that proof needs to be evaluated. This can be easily performed for the left-hand side of Eq. (10) that contains only two terms instead of the three terms on the right-hand side of Eq. (6), thereby resulting in only a minor modification of the calculations [15] giving the result $\langle D_3^{(i)}\rangle^2+\langle D_3^{(i+1)}\rangle^2\leq 2$. ■

These results for orthogonal observables can again be interpreted in terms of the space given by the coordinates $\mathcal{D}_3^{(i)}$ (for $i=1,2,3$). The same structure as in Fig. 1 then arises but with the different numerical bounds of Theorem 2. The fully separable states are confined to a cube with edge length $\sqrt{3}$ and the biseparable states in partition $j=1,2,3$ are confined to cuboids with size either $\sqrt{6}\times\sqrt{3}\times\sqrt{3}$, $\sqrt{3}\times\sqrt{6}\times\sqrt{3}$, or $\sqrt{3}\times\sqrt{3}\times\sqrt{6}$. Furthermore, all three-qubit states are in the intersection of firstly the cube with edge length $\sqrt{6}$, secondly of the three orthogonal cylinders with radius $\sqrt{2}$, and thirdly of the sphere with radius $\sqrt{3}$.

The corresponding $\mathcal{D}_3^{(i)}-\mathcal{D}_3^{(i+1)}$ plane is drawn in Fig. 2. Compared to the case where no restriction was made to orthogonal observables (cf., Fig. 1) we see that we can still distinguish the different kinds of biseparable states, but they can still not be distinguished from fully three-particle entangled states since both types of states still have the same maximum for $\langle\mathcal{D}_3^{(i)}\rangle$. Furthermore, the ratio of the different maxima of $\langle\mathcal{D}_3^{(i)}\rangle$ for fully separable and biseparable states is still the same, i.e., the ratio is $\sqrt{2}/1=(\sqrt{3/2})/(\sqrt{3/4})=\sqrt{2}$.

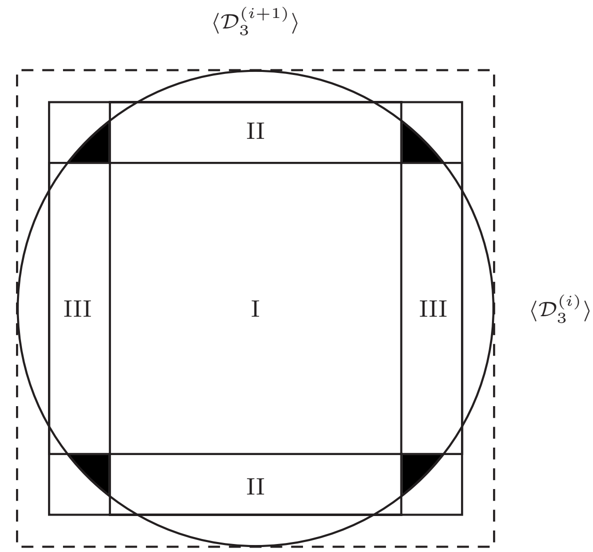

FIG. 2. The results of Theorem 3 for orthogonal observables. For comparison to the case where the observables were not restricted to be orthogonal, the dashed square is included that has edge length $2\sqrt{2}$ and which is the largest square in Fig. 1.

The black areas in Fig. 2 are again nonempty since states exist that have both $|\langle D_3^{(i)}\rangle|>\sqrt{3/4}$ and $|\langle D_3^{(i+1)}\rangle|>\sqrt{3/4}$ for the case of orthogonal observables. For example, the W-state of Eq. (8) gives $|\langle D_3^{(i)}\rangle|=0.906$ for all i, for the local angles $\alpha_i=0.54=\beta_i-\pi/2$ in the $x-z$ plane.

IV. DISCUSSION

Let us take another look at the quadratic inequalities $\langle D_3^{(i)}\rangle^2+\langle D_3^{(i+1)}\rangle^2\leq 5/2$ of Eq. (7) for general observables and $\langle D_3^{(i)}\rangle^2+\langle D_3^{(i+1)}\rangle^2\leq 2$ of Eq. (10) for orthogonal observables. These can be interpreted as monogamy inequalities for maximal biseparable three-particle quantum correlations (i.e., biseparable correlations that violate the inequalities maximally), since the inequalities show that a state that has maximal biseparable Bell correlations for a certain partition can not have it maximally for another partition. Indeed, when partition i gives $|\langle D_3^{(i)}\rangle|=\sqrt{2}$ it must be the case according to Eq. (7) that for the other two partitions both $|\langle D_3^{(i+1)}\rangle|\leq\sqrt{1/2}$ and $|\langle D_3^{(i+2)}\rangle|\leq\sqrt{1/2}$ must hold. The latter two must thus be nonmaximal as soon as the first type of biseparable correlation is maximal; and for the second inequality of Eq. (10) using orthogonal observables we get that when $|\langle D_3^{(i)}\rangle|=\sqrt{3/2}$ (this is maximal) it must be the case that both $|\langle D_3^{(i+1)}\rangle|\leq\sqrt{1/2}$ and $|\langle D_3^{(i+2)}\rangle|\leq\sqrt{1/2}$, which is nonmaximal.

From this we see that the first [i.e., Eq. (7) for general observables] is a stronger monogamy relationship than the second [i.e., Eq. (10) for orthogonal observables] since the tradeoff between how much the maximal value for $|\langle D_3^{(i)}\rangle|$ for one partition i restricts the value of $|\langle D_3^{(i+1)}\rangle|$, $|\langle D_3^{(i+2)}\rangle|$ for the other two partitions below the maximal value is larger in the first case than in the second case.

Let us see how this compares to the monogamy inequality $\langle\mathcal{B}_2^{(i)}\rangle^2+\langle\mathcal{B}_2^{(i+1)}\rangle^2\leq 2$ which was recently obtained by Toner and Verstraete [7]. Note that $|\langle \mathcal{B}_2^{(i)}\rangle| \leq 1$ is the ordinary Bell-CHSH inequality for the two qubits other than qubit i. We see that this monogamy inequality is even more strong than the ones presented here, since when $|\langle \mathcal{B}_2^{(i)}\rangle|$ obtains its maximal value of $\sqrt{2}$ it must be that $|\langle \mathcal{B}_2^{(i+1)}\rangle| = |\langle \mathcal{B}_2^{(i+2)}\rangle| = 0$.

Furthermore, the monogamy relationship of Toner and Verstraete shows that the so-called nonlocality that is indicated by correlations that violate the Bell-CHSH inequality [16] cannot be shared (cf. [12]): as soon as for some i one has $|\langle \mathcal{B}_2^{(i)}\rangle| > 1$, it must be that both $|\langle \mathcal{B}_2^{(i+1)}\rangle| < 1$ and $|\langle \mathcal{B}_2^{(i+2)}\rangle| < 1$. However, in Ref. [13] it was nevertheless shown that a bipartite Bell-type inequality exists where it is the case that the nonlocality that this inequality allows for can be shared. Since $|\langle D_3^{(i)}\rangle| \leq 1$ are Bell-type inequalities [i.e., local realism has to obey them, see Eq. (1)] whose violation can be seen to indicate some nonlocality, the inequalities considered here could possibly also allow for some nonlocality sharing.

Indeed, this is the case since it was shown that the black areas in Fig. 1 are nonempty. The Bell-type inequalities given here thus allow for sharing of the nonlocality of biseparable three-particle quantum correlations that is indicated by a violation of these inequalities.

In conclusion, we have presented stronger bounds for biseparable correlations in three-partite systems than were given in [2] and extended this analysis to the case of the restriction to orthogonal observables which gave even stronger bounds. The quadratic inequalities for biseparable correlations gave a monogamy relationship for correlations that violate the inequalities maximally (i.e., these cannot be shared), but they nevertheless did allow for sharing of the nonmaximally violating correlations.

We hope that future research will reveal more of the structure of the different kinds of partial separability in multipartite states and of the monogamy of multipartite Bell correlations. It could therefore be fruitful to generalize this work from three to a larger number of parties.

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[14] In further detail, steps (1)–(4) of the proof in Sun and Fei [2] become (using the terminology of their proof): (1) $\omega = 2(\boldsymbol{s}_1 \otimes \boldsymbol{s}_2 \otimes \boldsymbol{s}_3 \cdot \boldsymbol{Q})^2 = 2\langle \Psi | C_1 C_2 C_3 | \Psi\rangle^2 \leq 2$; (2) $\omega = 2(\boldsymbol{s}_1 \otimes \boldsymbol{s}_2 \otimes \boldsymbol{s}_3 \cdot \boldsymbol{Q} + \boldsymbol{s}_1 \otimes \boldsymbol{s}_2 \otimes \boldsymbol{t}_3 \cdot \boldsymbol{Q})^2 = 2\langle \Psi | C_1 C_2 (C_3 + D_3) | \Psi\rangle^2 \leq 2$; (3) $\omega = (5/4) \times [\cos(\theta_1 + \theta_2 + \theta_3) - \sin(\theta_1 + \theta_2 + \theta_3)]^2 \leq 5/2$; and (4) $\omega = [\cos(\theta_1 + \theta_2 + \theta_3) - \sin(\theta_1 + \theta_2 + \theta_3)]^2 \leq 2$. Here $\omega = \langle \mathcal{D}_3^{(i)}\rangle^2 + \langle \mathcal{D}_3^{(i+1)}\rangle^2$ [i.e., the left-hand side of Eq. (7)], where we have chosen $i=1$. Note that by symmetry the proof goes analogous for $i=2,3$. It follows that step (3) has the highest bound of $5/2$.

[15] In further detail, the proof in Sun and Fei [2] for the case of orthogonal observables amounts to (using the terminology of their proof) $|\boldsymbol{s}_i| = |\boldsymbol{t}_i| = \sqrt{2}/2$. Thus only step (4) needs to be evaluated and this gives $\omega = [\cos(\theta_1 + \theta_2 + \theta_3) - \sin(\theta_1 + \theta_2 + \theta_3)]^2 \leq 2$. As in the proof of Theorem 2 we have $\omega = \langle \mathcal{D}_3^{(i)}\rangle^2 + \langle \mathcal{D}_3^{(i+1)}\rangle^2$ [i.e., the left-hand side of Eq. (10)], where again we have chosen $i=1$, but by symmetry the proof goes analogous for $i=2,3$.

[16] A nonlocal correlation is a correlation that cannot be reproduced by shared randomness or any other local variables. It is detected by means of a violation of a Bell-type inequality.