Formality beyond Kähler geometry

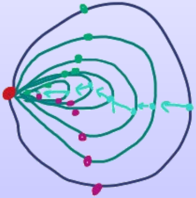
Gil Cavalcanti

Utrecht University

17 April 2013
Vrije Universiteit

The unanswerable question

What is $\pi_k(M)$?



What is $\pi_k(S^n)$?

Outline of Topics

1. Differential graded algebras
2. Minimal models and Sullivan's theorem
3. Kähler manifolds & beyond
4. The symplectic red herring

Differential graded algebras (DGAs)

Definition

A *DGA* is an $\mathbb{N}$-graded vector space $\mathcal{A}^\bullet$ endowed with

- a graded commutative product:

$$\mathcal{A}^k \times \mathcal{A}^l \longrightarrow \mathcal{A}^{k+l};$$

$$a \cdot b = (-1)^{|a||b|} b \cdot a.$$

- a degree 1 differential $d : \mathcal{A}^k \longrightarrow \mathcal{A}^{k+1}$, $d^2 = 0$;
- (Leibniz rule) $d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db)$.

Examples

- Given a manifold $(\Omega^\bullet(M), \wedge, d)$;
- Any graded algebra $(\mathcal{A}, \cdot)$ with the trivial differential $d := 0$;
- The cohomology algebra of a manifold $(H^\bullet(M;\mathbb{R}), \cup, 0)$.

Basics

Given a DGA $\mathcal{A}$ we can always form its cohomology

$$H^k(\mathcal{A}) = \frac{ker(d : \mathcal{A}^k \longrightarrow \mathcal{A}^{k+1})}{\mathrm{Im}\,(d : \mathcal{A}^{k-1} \longrightarrow \mathcal{A}^k)}$$

Definition

A DGA is *connected* if $H^0(\mathcal{A}) = \mathbb{R}$. A DGA is *1-connected* if it is connected and $H^1(\mathcal{A}) = \{0\}$.

Basics

- A map of DGAs is a linear map $f : \mathcal{A} \longrightarrow \mathcal{B}$ such that
  1. $f : \mathcal{A}^k \longrightarrow \mathcal{B}^k$;
  2. $f(a \cdot b) = f(a) \cdot f(b)$;
  3. $f \circ d = d \circ f$.
- A map of DGAs $f : \mathcal{A} \longrightarrow \mathcal{B}$ induces a map in cohomology

$$f^* : H^\bullet(\mathcal{A}) \longrightarrow H^\bullet(\mathcal{B}).$$

Definition

A *quasi isomorphism* is a map of DGAs $f : \mathcal{A} \longrightarrow \mathcal{B}$ which induces an isomorphism in cohomology.

Minimal DGAs

Definition

A DGA $\mathcal{M}$ is *minimal* if it is freely generated as a graded algebra and has an ordered set of generators $\{a_i\}_{i\in I}$ such that

1. $|a_i| \leq |a_j|$ if $i < j$;
2. $|a_i| > 0$;
3. $da_i \in \mathcal{A}^{<i} \wedge \mathcal{A}^{<i}$.

Example

Let $\mathcal{M} = \langle a_1 : |a_1| = 2n-1; da_1 = 0\rangle$

$$\varphi : \mathcal{M}^\bullet \longrightarrow \Omega^\bullet(S^{2n-1}), \qquad \varphi(a_1) = \sigma.$$

φ is a q.i.

Examples

Example

Let $\mathcal{M} = \langle a_1, a_2 : |a_1| = 2n; |a_2| = 4n-1; da_1 = 0; da_2 = a_1^2 \rangle$

$$\varphi : \mathcal{M}^\bullet \longrightarrow \Omega^\bullet(S^{2n}), \qquad \varphi(a_1) = \sigma, \;\; \varphi(a_2) = 0.$$

φ is a q.i.

Example

Let $\mathcal{M} = \langle a_1, a_2 : |a_1| = 2; |a_2| = 2n+1; da_1 = 0; da_2 = a_1^{n+1} \rangle$

$$\varphi : \mathcal{M}^\bullet \longrightarrow \Omega^\bullet(\mathbb{C}P^n), \qquad \varphi(a_1) = \omega, \;\; \varphi(a_2) = 0.$$

φ is a q.i.

Minimal models

Definition

A *minimal model* for a DGA $\mathcal{A}$ is

1. a minimal DGA $\mathcal{M}$;
2. a quasi isomorphism $\varphi : \mathcal{M} \longrightarrow \mathcal{A}$.

Examples 1–3 give minimal models for $\Omega(S^n)$ and $\Omega(\mathbb{C}P^n)$.

Theorem

Every 1-connected DGA has a minimal model.

Theorem (“The minimal model”)

If $\mathcal{A}$ and $\mathcal{B}$ are quasi-isomorphic, then their minimal models are isomorphic.

Minimal models

Remark

(Sullivan 74) The data needed to construct the minimal model of $\Omega(M)$ is precisely the same as that needed to construct the rational Postnikov tower of M.

Theorem (Sullivan 74)

Let M be a connected and simply connected manifold. Then

$$\pi_k(M) \otimes \mathbb{R} = span\{generators\ of\ \mathcal{M}\ of\ degree\ k\}$$

Minimal models

Corollary

- $\pi_k(S^{2n-1}) \otimes \mathbb{R} = \mathbb{R}$ *if* $k = 2n - 1$ *and zero otherwise;*
- $\pi_k(S^{2n}) \otimes \mathbb{R} = \mathbb{R}$ *if* $k = 2n$ *or* $4n - 1$ *and zero otherwise;*
- $\pi_k(\mathbb{C}P^n) \otimes \mathbb{R} = \mathbb{R}$ *if* $k = 2$ *or* $2n + 1$ *and zero otherwise;*

Definition

A manifold M is *formal* if $\Omega(M)$ and $H(M)$ have the same minimal model.

Minimal models

Example

In symmetric spaces (G/H) the product of harmonic forms is again harmonic, hence we have a quasi-isomorphism

$$H(M) \longrightarrow \Omega(M).$$

Hence all symmetric spaces are formal (e.g., spheres, $\mathbb{C}P^n$, grassmannians).

Example (Miller 79)

A compact k-connected manifold of dimension $\leq 4k + 2$ is formal.

Kähler manifolds

Theorem (DGMS 75)

Any compact Kähler manifold is formal.

key ingredient: dd^c-lemma

$$\operatorname{Im} d \cap \ker d^c = \operatorname{Im} d^c \cap \ker d = \operatorname{Im}(dd^c).$$

Proof.

Use the fact that d^c is a derivation and $dd^c = -d^c d$ to construct

$$(\Omega(M), d) \hookleftarrow (\Omega_{d^c}(M), d) \to (H_{d^c}(M), d).$$

Use five times the dd^c-lemma to prove that the maps are quasi-isomorphisms and the differential in the rightmost algebra is trivial. □

Kähler manifolds

Proof.

For example, to show that $\iota : \Omega_{d^c}(M) \hookrightarrow \Omega(M)$ induces an injection in cohomology.
Let $a \in H(\Omega_{d^c}(M))$ be such that $\iota^* a = 0$. Let $\alpha \in \Omega_{d^c}(M)$ be a rep.

$$\alpha \in \ker(d^c) \cap \mathrm{Im}\,(d) \Rightarrow \alpha \in \mathrm{Im}\,(dd^c) \Rightarrow \alpha = dd^c\beta$$

$d^c\beta \in \Omega_{d^c}(M)$ & $d(d^c\beta) = \alpha$ $\square$

Beyond Kähler

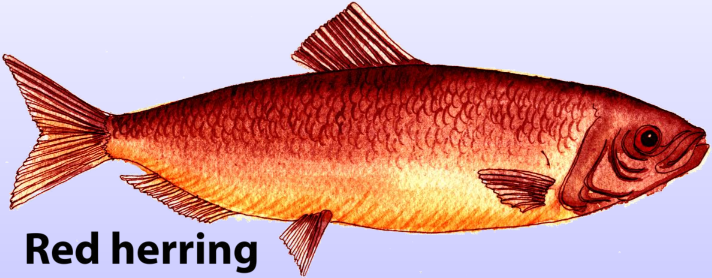
Possible extention: manifolds with reduced holonomy.

Berger's list of holonomy groups

Hol	name	formal
H	Symmetric spaces (G/H)	✓
$U(n)$	Kähler	✓
$SU(n)$	Calabi–Yau	✓
$Sp(n)$	Hyper-Kähler	✓
$Sp(n) \times Sp(1)$	Quaternionic Kähler	✓ (Amann 2009)
G_2	G_2	?
$Spin(7)$	$Spin(7)$	?

Extention: Generalized Kähler geometry (Cavalcanti 2007).

The symplectic red herring



Red herring

something, especially a clue that is or is intended to be misleading or distracting: *the book is fast paced, exciting and full of red herrings.*

The symplectic red herring

Question: Are compact symplectic manifolds formal?
This question is silly: Thurston's first example of nonKähler symplectic manifold (1974) is not formal.
Why? Because it has a nontrivial Massey product.

Massey products

Let $a_i \in H(\mathcal{A})$, $i = 1, 2, 3$ be such that $a_1 \cup a_2 = a_2 \cup a_3 = 0$.
Let α_i represent a_i and define $\alpha_{i,j}$ by the identities

$$d\alpha_{1,2} = \alpha_1 \wedge \alpha_2; \qquad d\alpha_{2,3} = \alpha_2 \wedge \alpha_3.$$

Then

$$\langle a_1, a_2, a_3 \rangle = [\alpha_{1,2} \wedge \alpha_3 + (-1)^{|a_1|+1} \alpha_1 \wedge \alpha_{2,3}] \in H(\mathcal{A})/\mathcal{I}(a_1, a_3).$$

The form

$$\alpha_{1,2} \wedge \alpha_3 + (-1)^{|a_1|+1} \alpha_1 \wedge \alpha_{2,3}$$

is closed for purely combinatorial reasons and the vanishing (or not) of its cohomology class is preserved by q.i.

Massey products

Theorem

If M is formal, all its Massey products vanish.

The symplectic red herring

Back to Thurston's example, $M = \mathbb{H} \times S^1$, where $\mathbb{H} = H^3/\Gamma$. And $\mathbb{H}$ has a nontrivial Massey product (this is a simple Lie algebra computation).

The symplectic red herring

Question (Lupton–Oprea 94)

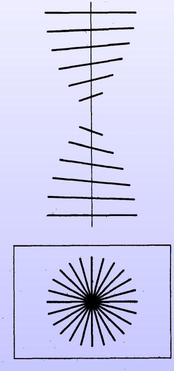
Are compact 1-connected symplectic manifolds formal?

This was also silly. In 94 there was only one source of 1-connected symplectic non Kähler manifolds (McDuff 84).

$$M \hookrightarrow \mathbb{C}P^n$$

symplectic blow-up: $\widetilde{\mathbb{C}P^n}$.

The symplectic red herring



Babenko–Taimanov (00) showed that McDuff's manifolds are not formal. For dimensions 10 and 8, examples were constructed by Fernandez–Muñoz in (02 & 05).

The symplectic red herring

In symplectic geometry the analogue of d^c is

$$\delta = \pi d - d\pi$$

Is there a $d\delta$-lemma?

The symplectic red herring

Theorem (Merkulov 98)

In a compact symplectic manifold (M^{2n}, ω) the following properties are equivalent:

1. *$d\delta$-lemma:*

$$\operatorname{Im} d \cap \ker \delta = \operatorname{Im} \delta \cap \ker d = \operatorname{Im} (d\delta);$$

2. *(hard) Lefschetz property:*

$$[\omega]^k : H^{n-k}(M) \longrightarrow H^{n+k}(M)$$

is an isomorphism for every k.

The symplectic red herring

Since δ is not a derivation, $\Omega_\delta(M)$ is not a DGA and formality does not follow.

Question (Babenko–Taimanov 00)

Does the Lefschetz property imply formality?

The symplectic red herring

Cohomology of $\tilde{X}$, the blow-up of X along $M^{2n-2k} \hookrightarrow X^{2n}$, was described by Porteous (60).

$$H(\tilde{X}) = H(X) + aH(M) + a^2H(M) + \cdots + a^{k-1}H(M); \quad |a| = 2.$$

with the relation

$$a^k = -c_1a^{k-1} - c_2a^{k-2} - \cdots - c_{k-1}a - t.$$

The symplectic red herring

Theorem

Let $\tilde{X}$ be the blow-up of X along $M^{2n-2k} \hookrightarrow X^{2n}$. Then

1. *(Babenko–Taimanov 00) If M has a nontrivial Massey product and $k > 3$, $\tilde{X}$ has a nontrivial Massey product;*
2. *(Cavalcanti 04) If X has a nontrivial Massey product, $\tilde{X}$ has a nontrivial Massey product.*

Roughly, if M or X are not formal, neither is $\tilde{X}$.

The symplectic red herring

(McDuff 84) The symplectic form on the blow-up represents the class $[\omega] + \varepsilon a$. Even if $\Sigma^2 \subset X^{2n}$ we can compute:

$$([\omega] + \varepsilon a)^{n-2} : H^2(X) \oplus aH^0(\Sigma) \longrightarrow H^{2n-2}(X) \oplus a^{n-2}H^2(\Sigma)$$

$$(\xi, \eta) \mapsto ([\omega]^{n-2}\xi - \varepsilon^{n-2}\eta t, \varepsilon^{n-3}(\sigma\eta + \varepsilon(\xi|_\Sigma - c_1\eta))),$$

This is a deformation of the "expected map"

$$(\xi, \eta) \mapsto ([\omega]^{n-2}\xi, \varepsilon^{n-3}\sigma\eta),$$

And it can have less kernel... can even become an isomorphism!

The symplectic red herring

In 2004, symplectic manifolds which were not formal and did not satisfy Lefschetz came a dime a dozen.

The symplectic red herring

Example (Cavalcanti 04)

$\mathbb{H} \times \mathbb{H}$ has a symplectic structure, a nontrivial Massey product and does not satisfy the Lefschetz property.
After blowing up a torus

$$T^2 \hookrightarrow \mathbb{H} \times \mathbb{H}$$

we get a manifold $\tilde{X}$ which *does* satisfy the Lefschetz property and has a nontrivial triple product.
This is a non simply connected counter-example to BT question.

The symplectic red herring

Example (Cavalcanti 04)

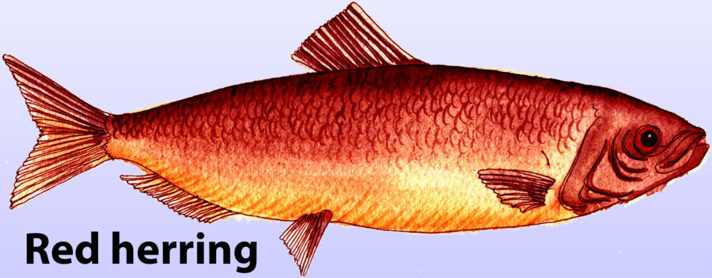
Embedding $\tilde{X} \hookrightarrow \mathbb{C}P^7$ and blowing up we get that $\widetilde{\mathbb{C}P}^7$ is a simply connected counter-example to the BT question.

The symplectic red herring

Theorem (Cavalcanti 04; Cavalcanti–Fernandez–Muñoz 08)

- *In any dimension the Lefschetz property is not related to formality.*
- *For 1-connected manifolds, in any dimension greater than 6 the Lefschetz property is not related to formality.*

The symplectic red herring



Red herring

something, especially a clue that is or is intended to be misleading or distracting: *the book is fast paced, exciting and full of red herrings.*