MORSE THEORY

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Contents

1. Lecture One. Propaganda. Poincaré conjecture in dimensions $\geqslant 5$.

1.1. **Singularities of smooth maps.** Let M, M' be smooth manifolds[1], and $f : M \longrightarrow M'$ a smooth map. We define

$$\mathrm{Crit}(f) := \{x \in M : \mathrm{rank}\, d_x f < \min(\dim M, \dim M')\}$$

the set of **critical points** of f, and $f\,\mathrm{Crit}(f) \subset M'$ the set of **critical values** of f. $M \diagdown \mathrm{Crit}(f)$ and $\mathbb{R} \diagdown f\,\mathrm{Crit}(f)$ are then said to consist of **regular points** and **regular values**, respectively.

Note that $\mathrm{Crit}(f) \subset M$ is closed.

1.1.1. *Abundance of regular values : Sard's theorem.* Recall that a subspace $X \subset \mathbb{R}^m$ is said to have **measure zero** if, for all $\varepsilon > 0$ there is a sequence of balls $(B_n)_{n \geqslant 0}$, with

$$\sum_n \mathrm{vol}(B_n) < \varepsilon, \quad \bigcup_n B_n \supset X. \quad \mathrm{vol}(B_n) := \int_{B_n} dx$$

One immediately checks that :

- $(X_n)_{n \geqslant 0}$ have measure zero $\Longrightarrow \bigcup_n X_n$ has measure zero;
- $g : \mathbb{R}^m \to \mathbb{R}^m$ smooth, $X \subset \mathbb{R}^m$ of measure zero $\Longrightarrow g(X) \subset \mathbb{R}^m$ has measure zero.

Hence the following notion is well-defined : a subspace $X \subset M$ is said to have measure zero if there exists a smooth atlas $\mathfrak{A} = \{(U_i, \varphi_i)\}$ of M, with each

$$\varphi_i(X \cap U_i) \subset \mathbb{R}^m$$

of measure zero.

Recall now :

Theorem 1 (Sard)**.** *If $f : M \longrightarrow M'$ is smooth, $f\,\mathrm{Crit}(f) \subset M'$ has measure zero.*

Proof. See [10] or [30]. □

Hence 'almost all' values are regular.

When $\dim M \ll \dim M'$, there is a 'lot of space' to deform f inside M', and we can always *remove* the singularities of f – i.e., perturb it slightly to a f' with $\mathrm{Crit}(f') = \varnothing$.

Theorem 2 (Whitney)**.** *Every $f : M \to M'$ is C^∞-close to a injective immersion if $\dim M' \geqslant 2 \dim M$, and every M^m embeds in $\mathbb{R}^{2m}$.*

In the other extreme, if M is compact, without boundary, $\mathrm{Crit}(f) \neq \varnothing$.

So we cannot get rid of singularities of functions.

1.1.2. *Singularities of Functions.* We consider the assignement $M \mapsto C^\infty(M)$.

> **Meta-principle 1 : Min-Max:** "The more complicated the topology of M, the greater the number of critical points of functions on it."

Example 1. *If $f : \mathbb{T}^n \to \mathbb{R}$, then $|\mathrm{Crit}(f)| \geqslant n + 1$.*

For more, see Min-Max theory ⊛.

> **Meta-principle 2 : Morse-Smale:** "The dynamics of a nice enough $f \in C^\infty(M)$ reconstructs M smoothly."

Example 2. *Suppose M^m is compact without boundary, and $f : M \to \mathbb{R}$ has exactly two critical points. Then M^m is* homeomorphic *to S^m.*

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[1] Throughout these notes, by a *manifold*, we mean a Hausdorff, second-countable topological space, equipped with a maximal smooth atlas.

[DRAWING ⊛]

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Aside : Poincaré Conjecture & Homotopy Spheres

Remark 1. *It is* not *claimed that M^m is diffeomorphic to S^m, with its standard smooth structure. In fact, in* [15]*, Milnor constructs smooth $\mathbb{S}^3$-bundles $p : M \to \mathbb{S}^4$, for which there cannot exist B^8 with*

$$\partial B = M, \quad H^4(B; \mathbb{Z}) = 0,$$

and carries a smooth $f \in C^\infty(M)$ with exactly two non-degenerate critical points. This implies that M^7 is homeomorphic to $\mathbb{S}^7$, but not diffeomorphic to it; such manifolds are called **exotic spheres***.*

Definition 1. *A* **homotopy sphere** *is a smooth, oriented manifold M^m, homotopy-equivalent to $\mathbb{S}^m$.*

Note that if M^m is a homotopy sphere, then $\pi_1(M) = \{1\}$, and $H_\bullet(M; \mathbb{Z}) \simeq H_\bullet(\mathbb{S}^m; \mathbb{Z})$. Conversely, if M^m is simply connected and $H_\bullet(M; \mathbb{Z}) \simeq H_\bullet(\mathbb{S}^m; \mathbb{Z})$, then M^m is a homotopy sphere; indeed, in that case $\pi_\bullet(M) \simeq \pi_\bullet(\mathbb{S}^m)$ by Hurewicz' theorem. Now, a generator $[\alpha] \in \pi_m(\mathbb{S}^m)$, $\alpha : \mathbb{S}^m \to M$, gives rise to a homotopy equivalence $\mathbb{S}^m \xrightarrow{\sim} M$.

Homotopy spheres are the object of the famous

Theorem 3 (Poincaré Conjecture)**.** *A homotopy sphere M^m is homeomorphic to $\mathbb{S}^m$.*

Observe that the *smooth* version of the theorem, claiming that homotopy spheres are *diffeomorphic* to $\mathbb{S}^m$, is decidedly false in light of the existence of exotic spheres.

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Example 3. *Milnor's stand-up torus... ⊛. How does one make a drawing ?*

- *Present $(\mathbb{T}^2, f)$ and its 4 critical points;*
- *'Non-degenerate' allows* normal forms *around the points; describe them;*
- *Show how the topology of $f^{-1}(-\infty, t]$ changes as t varies. ('reconstruction').*

Note that :

(1) If $[a, b]$ contains no critical values of f, then the diffeomorphism type of $f^{-1}(-\infty, t]$ is independent of $t \in [a, b]$;

(2) When t crosses a critical value $t = c$, we have

$$f^{-1}(-\infty, c + \varepsilon] = f^{-1}(-\infty, c - \varepsilon] \cup H_\lambda,$$

where H_λ denotes a "handle" $H_\lambda \sim \mathbb{D}^\lambda \times \mathbb{D}^{m-\lambda}$.

In view of Sard's theorem, (1) suggests that we subdivide our task of understanding the topology of M.

Definition 2. *A* **cobordism** $\mathcal{C} = (W; M_0, f_0, M_1, f_1)$ *from M_0^m to M_1^m is a smooth manifold W^{m+1}, together with a decomposition of its boundary as $\partial W = \partial_0 W \coprod \partial_1 W$, together with diffeomorphisms $f_i : \partial_i W \xrightarrow{\sim} M_i$.*

If M_i are oriented (as will usually be the case), we assume further that W is oriented, and that f_0 be orientation-*preserving*, while f_1 is orientation-*reversing*; we refer to $\mathcal{C}$ as an **oriented cobordism** between M_0 and M_1. We will also refer to $\partial_0 W$ as the **incoming** boundary of W, and to $\partial_1 W$ as the **outgoing** boundary of W.

We regard a cobordism $\mathcal{C}$ as a 'morphism' $M_0 \rightsquigarrow M_1$ of sorts. The maps will be suppressed from the notation when no confusion can arise.

Definition 3. *Two cobordisms $\mathcal{C} = (W, M_0, f_0, M_1, f_1)$ and $\mathcal{C} = (W', M_0', f_0', M_1', f_1')$ are said to be* **equivalent over** M_0 *if there exists an oriented diffeomorphism*

$$F : W \xrightarrow{\simeq} W', \quad f_0' \circ F = f_0.$$

A cobordism $\mathcal{C} = (W, M_0, f_0, M_1, f_1)$ is called :

- **trivial** *if it is equivalent over M_0 to $(M_0 \times [0,1]; M_0, M_1)$;*
- *an h-***cobordism** *if $\partial_i W \hookrightarrow W$ are homotopy equivalences.*[2]

One of the goals of this course is to prove the following fundamental

Theorem 4 (Smale's h-cobordism theorem)**.** *If $\pi_1 M_0 = \{1\}$ and $\dim M \geqslant 5$, any h-cobordism over M_0 is trivial.*

Corollary 1 (Characterization of disks)**.** *If M^m is a contractible, smooth, compact manifold, and $\pi_1(\partial M) = \{1\}$, then $M \simeq D^m$ if $m \geqslant 6$.*

Proof. Choose an embedding $j : \mathbb{D}^m \hookrightarrow M^m$, and let

$$\widehat{M} := M \diagdown j(\mathring{\mathbb{D}}^m).$$

Then $j : \partial\mathbb{D}^m \hookrightarrow \widehat{M}$ induces a long exact sequence

$$\cdots \to H_{k+1}(\widehat{M}, j(\partial\mathbb{D}^m); \mathbb{Z}) \to H_k(j(\partial\mathbb{D}^m); \mathbb{Z}) \to H_k(\widehat{M}; \mathbb{Z}) \to H_k(\widehat{M}, j(\partial\mathbb{D}^m); \mathbb{Z}) \to \cdots$$

But note that $M \sim \widehat{M}/j(\partial\mathbb{D}^m)$, so

$$H_\bullet(\widehat{M}, j(\partial\mathbb{D}^m); \mathbb{Z}) = H_\bullet(\widehat{M}/j(\partial\mathbb{D}^m; \mathbb{Z}) = H_\bullet(M; \mathbb{Z}).$$

By hypothesis, $H_\bullet(M; \mathbb{Z}) = 0$, so we conclude that $H_\bullet(j_\partial) : H_\bullet(j(\partial\mathbb{D}^m); \mathbb{Z}) \to H_\bullet(\widehat{M}; \mathbb{Z})$ is an isomorphism. Hence by Whitehead's theorem, and the fact that $\pi_1(\widehat{M}) = \{1\}$, we see that $j_\partial : \partial\mathbb{D}^m \hookrightarrow \widehat{M}$ is a homotopy equivalence. Thus $(\widehat{M}; \partial\mathbb{D}^m, \partial M)$ is an h-cobordism.

By Smale's theorem, there is an equivalence

$$F : \widehat{M} \xrightarrow{\simeq} \partial\mathbb{D}^m \times [0,1]; \partial\mathbb{D}^m \times \{0\}, \partial\mathbb{D}^m \times \{1\}).$$

But M is clearly recovered as

$$\begin{tikzcd} \partial\mathbb{D}^m \arrow[r, "j_\partial"] \arrow[d, hook] & \widehat{M} \arrow[d, dashed] \\ \mathbb{D}^m \arrow[r, dashed] & M \end{tikzcd}$$

so $M \simeq (\mathbb{S}^{m-1} \times [0,1]) \cup_{\mathbb{S}^{m-1}} \mathbb{D}^m \simeq \mathbb{D}^m$. $\square$

Corollary 2 (Poincaré conjecture in high dimensions)**.** *M^m homotopy sphere, $m \geqslant 6 \Longrightarrow M$ is homeomorphic to $\mathbb{S}^m$.*

Proof. As before, start with an embedding $j : \mathbb{D}^m \hookrightarrow M^m$, and let $\widehat{M} := M \diagdown j(\mathring{\mathbb{D}}^m)$, so that $H_k(\widehat{M}, j(\mathbb{D}^m); \mathbb{Z}) = H_k(M; \mathbb{Z})$ still holds. The long exact sequence of the pair $(\widehat{M}, j(\partial\mathbb{D}^m))$ implies that

$$H_k(j(\mathbb{D}^m); \mathbb{Z}) = H_k(\widehat{M}; \mathbb{Z}), \quad k \leqslant n-2,$$

since $H_k(M; \mathbb{Z}) = H_k(\mathbb{S}^m; \mathbb{Z}) = 0$ for $k \neq 0, m$. For the case $k = m-1$ we have

$$0 \to H_m(M); \mathbb{Z}) \to H_{m-1}(j(\mathbb{D}^m); \mathbb{Z}) \to H_{m-1}(\widehat{M}; \mathbb{Z}) \to 0;$$

[2] Note that this is the homotopy-theoretic version of (1) above.

but note that maps the fundamental class of M to that of $j(\partial\mathbb{D}^m)$:

$$H_m(M);\mathbb{Z}) \ni [M] \mapsto [j(\mathbb{D}^m)] \in H_{m-1}(j(\mathbb{D}^m);\mathbb{Z}),$$

and thus $H_{m-1}(\widehat{M};\mathbb{Z}) = 0$.

Hence $\pi_1(\partial\widehat{M}) = \{1\}, \quad H_\bullet(\widehat{M};\mathbb{Z}) = 0$, so by Corollary 1, there is a diffeomorphism $i : \mathbb{D}^m \xrightarrow{\simeq} \widehat{M}$.

Consider the diffeomorphism $f := j \circ i^{-1} : i(\partial\mathbb{D}^m) \xrightarrow{\simeq} j(\partial\mathbb{D}^m)$. In general, it is *not* possible to extend f to a diffeomorphism

$$\widetilde{f} : i(\mathbb{D}^m) \xrightarrow{\simeq} j(\mathbb{D}^m);$$

however, we can extend f to a *homeomorphism* $\widetilde{f} : i(\mathbb{D}^m) \xrightarrow{\sim} j(\mathbb{D}^m)$ by the so-called 'Alexander trick' :

$$\widetilde{f} : i(x) \mapsto |x| f\left(\frac{x}{|x|}\right).$$

We can now define a homeomorphism $F : \mathbb{S}^m \to M^m$ by

$$\begin{tikzcd}
\mathbb{D}^m_- \arrow[r, hook] \arrow[dr, "\widehat{f}\circ i"'] & \mathbb{S}^m \arrow[d, dashed, "F"', "\sim"] & \mathbb{D}^m_+ \arrow[l, hook'] \arrow[dl, "j"] \\
& M &
\end{tikzcd}$$

□

1.2. **Exercises.**

(1) Recall the definition of the weak and strong topologies in the function spaces $C^k(M, M')$, and that $C^r_W(M, M')$ has a complete metric.

(2) Show that $\mathrm{Prop}(M, M') \subset C^0_S(M, M')$ is a connected component.

(3) Let $U \subset M$ be open. The restriction map $C^r(M, M') \to C^r(U, M')$, $0 \leqslant r \leqslant \infty$, is continuous for the weak topology, but not always the strong. However, it is an open map for the strong topologies, and not always for the weak topology.

(4) A submanifold $X \subset W$ of a manifold with boundary is called **neat** if $\partial X = X \cap \partial W$, and X is not tangent to ∂W at any point $x \in \partial X$. Show that if $y \in M$ is a regular value for $f : W \to M$ and $f|_{\partial W} : \partial W \to M$, then $f^{-1}(y) \subset W$ is a neat submanifold.

(5) If $f : M \longrightarrow M'$ is smooth, and $X \subset M'$ is a submanifold, we say that f is **transverse** to X, written $f \pitchfork X$, if

$$\operatorname{im} d_x f + T_{f(x)} X = T_{f(x)} M', \quad x \in f^{-1}(X).$$

Suppose now that W is a manifold with boundary, and $f : W \to M'$ is smooth. If $f, f|_{\partial W} \pitchfork X$, then $f^{-1}X \subset W$ is a neat submanifold, and $\operatorname{codim}(f^{-1}X \subset W) = \operatorname{codim}(X \subset M')$.

(6) Every closed subpace $X \subset M$ can be described as $X = f^{-1}(0)$, where $f : M \to \mathbb{R}$ is a smooth function.

(7) Can you find a smooth $f : \mathbb{T}^2 \to \mathbb{R}$ with exactly three critical points ?

(8) Show that if W if a compact manifold with boundary, there can be no continuous map $r : W \to \partial W$ extending $\mathrm{id}_{\partial W}$.

2. Lecture Two. Normal Forms of smooth maps. Morse functions.

A general goal of this course is to understand how to extract information about the topology of M by means 'good' functions $f : M \to \mathbb{R}$.

We will be mostly concerned with *compact* manifolds without boundary, but many natural constructions lead us away from this more manageable case. When M is non-compact, we will typically demand that $f : M \to \mathbb{R}$ be **proper**.

Aside : proper maps

Recall that a map $f : M \to M'$ is said to be proper if f^{-1} takes compact sets to compact sets. The subspace $\mathrm{Prop}^k(M, M') \subset C^k(M, M')$ of proper maps is open in the strong C^k-topology, for every $k \geqslant 0$.

Exercise : if f is proper, then $f\,\mathrm{Crit}(f) \subset M'$ is closed.

One very strong reason to deal exclusively with proper maps is that non-proper maps may not reflect any of the topology of M. As an illustration, let us convene that an **open manifold** is a manifold, none of whose connected components is compact without boundary. Then we have

Theorem 5 (Gromov)**.** *On every open manifold M, there is $f \in C^\infty(M)$ with* $\mathrm{Crit}(f) = \varnothing$*.*

The catch is that such f cannot be proper.

Let us go back to our $f : M \to \mathbb{R}$ (proper if M is non-compact). We inaugurate the notation

$$M_t := f^{-1}(-\infty, t] \subset M.$$

Exercise : this is a smooth manifold with boundary $\partial M_t = f^{-1}(t)$ when t is a regular value for f.

Assume now that $[a, b] \subset \mathbb{R}$ consists only of regular values for f. Our first goal in this lecture is to prove the

Theorem 6 (Structure Theorem I)**.** *M_b is diffeomorphic to M_a, and the inclusion $M_a \hookrightarrow M_b$ is a strong deformation retraction.*

Before we give the proof, a short reminder comes in handy.

Aside : Vector fields and their flows

Recall that, by the Fundamental Theorem of ODEs, a vector field $w \in \mathfrak{X}(M)$ defines a **local flow**. That is, there is

$$\phi : M \times \mathbb{R} \supset \mathrm{dom}(\phi) \longrightarrow M,$$

where $\mathrm{dom}(\phi)$ is an open containing $M \times \{0\}$, with the property that, for each $x \in M$, $c(t) := \phi^t(x)$ is the maximal trajectory of w with initial condition $c(0) = x$. Being a trajectory of w means that $\frac{dc}{dt} = w \circ c$; by 'maximal trajectory' we mean that

$$c : \mathrm{dom}(\phi) \cap \{x\} \times \mathbb{R} =: (a_x, b_x) \to M$$

cannot be extended any further.

Note that $\phi^s(\phi^x(x)) = \phi^{t+s}(x)$ whenever either side of the equation is defined.

When $\mathrm{dom}(\phi) = M \times \mathbb{R}$, we say that ϕ is the **flow** of w; in this case, ϕ determines a group homomorphism $\phi : (\mathbb{R}, +) \to (\mathrm{Diff}(M), \circ)$, and we will say that w is **complete**. **Exercise :** w is complete if it is compactly supported.

However,

Example 4. *Neither $\partial/\partial t \in \mathfrak{X}(\mathbb{R}\setminus 0)$ nor $(1+t^2)\partial/\partial t \in \mathfrak{X}(\mathbb{R})$ are complete*[3].

There is a classical condition to be imposed on w to ensure that it give rise to a flow.

Definition 4. *A Riemannian metric g on a manifold M is called* **complete** *if the geodesics of its Levi-Civita connection are defined at all times.*

Complete Riemannian metrics exist on all manifolds of finite dimension.

Definition 5. *A vector field $w \in \mathfrak{X}(M)$ is said to have* **bounded velocity** *if there exists a* complete *Riemannian metric g on M, for which $\|w\|$ is bounded by some real number K :*

$$\sup_{x\in M} \|w_x\| \leqslant K < +\infty.$$

Lemma 1. *Let (M, g) be complete.*

(1) $(a, b) \subset \mathbb{R}$ a bounded interval, and $c : (a, b) \to M$ a curve of finite length :

$$\int_a^b \|c'(t)\|dt < \infty.$$

Then $\operatorname{im} c \subset M$ is precompact.

(2) Suppose $c(t)$ is a maximal trajectory of $w \in \mathfrak{X}(M)$, $c : J \to M$, where $J \subset \mathbb{R}$ is an interval containing 0. Then :
- $[0, +\infty) \nsubseteq J \Longrightarrow \int_0^b \|c'(t)\|dt = \infty$;
- $(-\infty, 0] \nsubseteq J \Longrightarrow \int_a^0 \|c'(t)\|dt = \infty$;

Proof. **(1) :** It suffices[4] to show that, for every $\varepsilon > 0$, there exist $x_0, ..., x_N \in \operatorname{Cl}\operatorname{im} c$ such that the ε-balls around x_i cover it : $\cup_1^N B_\varepsilon(x_i) \supset \operatorname{Cl}\operatorname{im} c$. But

$$\int_a^b \|c'(t)\|dt < \infty \quad \Longrightarrow \quad \exists a = t_0 < t_1 < \cdots < t_N = b, \quad \int_{t_i}^{t_{i+1}} \|c'(t)\|dt < \varepsilon,$$

so

$$\operatorname{Cl}(\operatorname{im} c) \subset \bigcup_0^N B_\varepsilon(c(t_i)).$$

(2) : If c is maximal, and $[0, \infty) \nsubseteq J$, then $c(t)$ has no limit point as $t \to b$, $b := \sup\{t : t \in J\} < \infty$. It then follows from the first part of the lemma that $\int_0^b \|c'(t)\|dt = \infty$. The other case is completely analogous. ☐

Definition 6. *An* **isotopy** *ψ of a smooth manifold M is a smooth map*

$$\psi : M \times J \to M,$$

where $J \subset \mathbb{R}$ is an interval containing 0, each $\psi_t := \psi(\cdot, t) : M \to M$ is a diffeomorphism, and $\psi_0 = \mathrm{id}_M$.

[3] For the second example, note that a solution curve $c(t)$ to $(1+t^2)\partial/\partial t$ with initial condition $c(0) = 0$ is $c(t) = \tan t$, which cannot be extended beyond $(-\pi/2, +\pi/2)$.

[4] A metric space (X, d) is called **totally bounded** if, for every $\varepsilon > 0$, X can be covered by finitely many ε-balls. A complete metric space is compact iff it is totally bounded. Indeed, it is clear that any compact space is totally bounded. On the other hand, if a space is totally bounded, to show that it is compact it is enough to show that every sequence $(x_n)_{n\geqslant 0}$ has a Cauchy subsequence $(x_{n_k})_{k\geqslant 0}$. Cover X with finitely many balls $B_1, ..., B_N$ of radius 1; then one of the balls, say B_1, must contain infinitely many terms of (x_n). This defines a subsequence $s_1 \subset (x_n)$, and the distance between any two points in s_1 is no greater than 1. Now cover B_1 by finitely many balls of radius 1/2; again, we can select a subsequence $s_2 \subset s_1$ of points lying in one single 1/2-ball. Inductively, we define then a sequence of subsequences $(x_n) \supset \cdots \supset s_k \supset s_{k+1} \supset \cdots$, with each s_k lying in a ball of radius $1/k$; hence a sequence $x_{n_k} \in s_k \setminus s_{k_1}$ must be Cauchy.

The flow of a complete vector field is an example of an isotopy.

An isotopy ψ gives rise to a **time-dependent vector field**, i.e., a one-parameter family $t \mapsto w_t$ of vector fields on M, defined by

$$\frac{d\psi_t}{dt} = w_t \circ \psi_t, \quad t \in J.$$

Remark 2. *A time-dependent vector field w_t can of course be regarded as an autonomous vector field $\widetilde{w}$ on $M \times J$, to which the above discussion applies to produce a local flow*

$$\widetilde{\phi} : (M \times J) \times \mathbb{R} \supset \mathrm{dom}(\widetilde{\phi}) \to M \times J.$$

Note however that $\widetilde{\phi}$ gives rise to an isotopy of $M \times J$, not of M alone. This can be remedied as follows : consider the autonomous

$$\widehat{w} := w_t + \partial/\partial t \in \mathfrak{X}(M \times J)$$

Let us assume for simplicity that $\widehat{w}$ is complete, and let us denote its flow by $\widehat{\phi}$: $(M \times J) \times \mathbb{R} \to M \times J$. Then $\widehat{\phi}$ satisfies

$$\widehat{\phi}_s(x,t) \in M \times \{t+s\},$$

so $\widehat{\phi}_{r+s}(x,t) = \widehat{\phi}_r \circ \widehat{\phi}_s(x,t)$ wherever this makes sense; in particular, $s \mapsto \mathrm{pr}_M \circ \widehat{\phi}_s$ gives rise to an isotopy of M.

We conclude that :

Lemma 2. *Time-dependent vector fields of bounded velocity give rise to isotopies.*

Proof. □

We conclude this aside by recalling a very useful formula from Calculus. Suppose ψ is an isotopy of M with corresponding time-dependent vector field $w_t \in \mathfrak{X}(M)$. Suppose $t \mapsto \eta_t$ is a time-dependent section of some tensor bundle $E := (\bigwedge^p TM) \otimes (\bigwedge^q T^*M)$.

Lemma 3. $\frac{d}{dt}(\psi_t^*\eta_t) = \psi_t^*\left(L(w_t)\eta_t + \frac{d\eta_t}{dt}\right)$.

Exercise : Prove the Lemma.

Proof of Structure Theorem I. Suppose

$$f\,\mathrm{Crit}(f) \cap [a,b] = \varnothing.$$

Then $f\,\mathrm{Crit}(f) \cap [a-\varepsilon, b+\varepsilon] = \varnothing$ for small enough $\varepsilon > 0$. Choose

$$\varrho : [a-\varepsilon, b+\varepsilon] \to [0,1]$$

such that

$$\varrho(t) = \begin{cases} 1 & \text{if } t \in [a-\varepsilon/3, b+\varepsilon/3]; \\ 0 & \text{if } t \notin [a-2\varepsilon/3, b+2\varepsilon/3]. \end{cases}$$

Let

$$w := \frac{-(\varrho \circ f)}{\|\nabla f\|^2}\nabla f \in \mathfrak{X}(M),$$

where $\|\cdot\|$ refers to some auxiliary (complete) Riemannian metric g on M and ∇f denotes the vector field defined by $g(\nabla f, v) = df(v)$.

Observe that

$$(L(w)f)(x) = \begin{cases} -1 & \text{if } f(x) \in [a-\varepsilon/3, b+\varepsilon/3]; \\ 0 & \text{if } f(x) \notin [a-2\varepsilon/3, b+2\varepsilon/3]. \end{cases}$$

f being proper, w is compactly supported, and so gives rise to a flow

$$\phi : M_b \times \mathbb{R} \longrightarrow M_b, \quad \phi_t(M_b) \subset M_{b-t}.$$

In particular, we have a diffeomorphism

$$\phi_{b-a} : M_b \xrightarrow{\simeq} M_a,$$

and

$$\phi|_{M_b \times [0,b-a]} : M_b \times [0, b-a] \longrightarrow M_b$$

is a strong deformation retraction of M_b onto M_a. □

This idea that 'in the absence of critical points we can push down M_t' can be turned around to *detect* critical points of a $f \in C^\infty(M)$.

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Aside : Palais-Smale Condition C
Fix a complete Riemannian manifold (M, g), and let $f : M \to \mathbb{R}$ be given.

Definition 7. *We say that f satisfies* **Condition C** *if, whenever a sequence $(x_n)_{n\geqslant 0}$ in M is such that*

- *$(|f(x_n)|)_{n\geqslant 0} \subset \mathbb{R}$ is bounded, and*
- *$\|\nabla f(x_n)\| \to 0$ as $n \to \infty$,*

then there is a subsequence $(x_{n_k})_{k\geq 0})$ converging in M.

Observe that any proper f satisfies Condition C automatically.

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Lemma 4. *Suppose f is bounded below, and f sastisfies Condition C. Then the flow ϕ^t of $-\nabla f$ is defined for all positive times, and for every $x \in M$, $\lim_{t\to+\infty} \phi^t(x)$ exists and is a critical point of f.*

Proof. Let $B := \inf_{x\in M} f(x) > -\infty$, and consider the maximal trajectory

$$c(t) := \phi^t(x), \quad c : J \to M;$$

we wish to show that $[0, +\infty) \subset J$.
First define $F : (a, b) \to \mathbb{R}$ by $F(t) := f(c(t))$. Then

$$B \leqslant F(t) = F(0) + \int_0^t F'(s)ds = F(0) - \int_0^t \|\nabla f(c(s))\|^2 ds$$
$$\Longrightarrow \quad \int_0^t \|\nabla f(c(s))\|^2 ds \leqslant F(0) - B.$$

Since the RHS is independent of t, we conclude that

$$\int_0^b \|\nabla f(c(s))\|^2 ds \leqslant F(0) - B.$$

Let us argue by contradiction, and assume that b were finite. By Schwarz's inequality,

$$\int_0^b \|\nabla f(c(s))\|ds \leqslant \sqrt{\int_0^b ds}\sqrt{\int_0^b \|\nabla f(c(s))\|^2 ds} \leqslant \sqrt{b(F(0) - B)}.$$

This implies that $\int_0^b \|\nabla f(c(s))\|ds < +\infty$. But by Lemma 1, $b < +\infty$ implies that $\int_0^b \|\nabla f(c(s))\|ds$ is infinite; the contradiction shows that $b = +\infty$.

But then

$$\int_0^\infty \|\nabla f(c(s))\|^2 ds \leqslant F(0) - B \quad \Longrightarrow \quad \|\nabla f_{c(t)}\|^2 \to 0, \text{ as } t \to \infty,$$

so $\|\nabla f_{c(t)}\| \to 0$. By Condition C, we can find $(t_n)_{n \geqslant 0}$ with $c(t_n) \to x \in M$; by continuity of df, we have

$$x \in \mathrm{Crit}(f).$$

□

We will return to this sort of argument in more detail when we deal with Min-Max theory.

2.1. **Normal Forms.** Having dealt with the regular case, we wish to understand the behavior of f around its *singular* points $x \in \mathrm{Crit}(f)$. Ideally, we should be able to provide a *model* for f around each critical point, depending only on the value of a (a priori known) finite number of derivatives of f at x.

For too badly behaved f, this is way too ambitious.

Example 5. *The maps $f_0, f_1 : \mathbb{R} \to \mathbb{R}$, $f_0(t) = 0$, and*

$$f_1(t) = \begin{cases} e^{-1/t} & \text{if } t \geqslant 0, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

both have 0 *as a critical point, and their derivatives at* 0 *vanish to infinite order, and they behave quite differently at zero.*

To weed out such behavior, and still hope to model the singularities of f, we should impose some *non-degeneracy* condition on the critical points $x \in \mathrm{Crit}(f)$.

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Aside : Germs

Recall that if M, M' are smooth manifolds, and $X \subset M$ is any subspace, we denote by

$$C^\infty(M, M')_X = \{[U, f] : X \subset U \subset M \text{ open}, f \in C^\infty(U, M')\},$$

where $[U, f]$ denotes the **germ** of f along X :

$$[U, f] = [U', f'] \quad \Longleftrightarrow \quad \exists U'' \subset U \cap U', \quad f|_{U''} = f'|_{U''}.$$

Two germs $[U, f], [U', f'] \in C^\infty(M, M')_X$ will be called **equivalent** if there exist $U'' \subset U \cap U'$, $V \supset f(X)$ opens, and embeddings $j : U'' \hookrightarrow U$ and $i : V \hookrightarrow M'$, with

$$if|_{U''} = f'|_{U''} j.$$

An equivalence class of germs around X formalizes the notion of 'behavior' around X : two maps $f, f' \in C^\infty(M, M')$ have **the same behavior** around $X \subset M$ iff their germs along X are equivalent.

We will typically be lazy, and write $[f]$ (or just f) instead of $[U, f]$.

We will mostly be concerned with $\mathcal{E} := \mathbb{C}^\infty(\mathbb{R}^m, \mathbb{R})_0$, the set of germs of real functions around zero. Note that this is a *ring*, with the operations

$$[f] + [f'] := [f + f'], \quad [f] \cdot [f'] := [ff'],$$

with additive and multiplicative units [0] and [1] respectively. Observe that $\mathcal{E}$ comes equipped with a natural surjective ring homomorphism

$$\mathrm{ev} : \mathcal{E} \to \mathbb{R}, [f] \mapsto f(0).$$

Since $\mathcal{E}/\mathbb{R}$ is a field, $\mathfrak{m} := \ker(\mathrm{ev})$ is a maximal ideal in $\mathcal{E}$; observe that $[f] \notin \mathfrak{m}$ implies that $[f]^{-1} = [f^{-1}] \in \mathcal{E}$, so $\mathfrak{m} \lhd \mathcal{E}$ is the *unique* maximal ideal – that is, $\mathcal{E}$ is a local ring.

Observe that $[f] \in \mathfrak{m}$ iff

$$f(x) = \int_0^1 \frac{d}{dt}(f(tx))dt = \sum_1^m \left(\int_0^1 \frac{\partial f}{\partial x_i}(tx) \right) \cdot x_i,$$

so $\mathfrak{m} = \sum_1^m \mathcal{E} \cdot x_i$; in particular, $\mathfrak{m}^2 = \sum_1^m \mathcal{E} \cdot x_i x_j$ and thus $[f] \in \mathfrak{m}^2$ iff $0 \in \mathrm{Crit}(f)$. This implies that

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} T_0^*\mathbb{R}^m, \quad [f] + \mathfrak{m}^2 \mapsto d_0 f,$$

is an isomorphism of $\mathcal{E}$-modules.

This observation can be expanded by observing that ev extends to a ring homomorphism

$$\mathrm{Tayl} : \mathcal{E} \to \mathbb{R}[[x_1, ..., x_m]], \quad [f] \mapsto \mathrm{Tayl}(f) := \sum_\alpha \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x_\alpha} x^\alpha,$$

where for a multi-index $\alpha = (\alpha_1, ..., \alpha_m)$, $\alpha_i \geqslant 0$, we set

$$|\alpha| := \sum \alpha_i, \quad \alpha! := \prod_1^m \alpha_i!, \quad x^\alpha = \prod_1^m x_i^{\alpha_i}, \quad \frac{\partial^{|\alpha|}}{\partial x_\alpha} := \prod_1^m \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}.$$

The homogeneous part of degree k of $\mathrm{Tayl}(f)$, denoted by $\mathrm{Tayl}^k(f)$, can be described in a slightly less coordinate-dependent fashion. Indeed, if $f : \mathbb{R}^m \to \mathbb{R}$ is a smooth map, then df can be regarded as a smooth map $df : \mathbb{R}^m \to \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}) \simeq \mathbb{R}^m$, and as such we can take $d(df) := d^2 f : \mathbb{R}^m \to \mathrm{Hom}(\mathbb{R}^m, \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}))$. But recall from Calculus that $d^2 f$ lands inside $\mathrm{Hom}^2(\mathbb{R}^m, \mathbb{R})$, i.e., $d^2 f(v, w)$ is symmetric in its arguments $v, w \in T_0\mathbb{R}^m$. More generally, we denote by $d^k f$ the map $d(d^{k-1} f) : \mathbb{R}^m \to \mathrm{Hom}^k(\mathbb{R}^m, \mathbb{R})$; in this notation,

$$\mathrm{Tayl}^k(f) = \frac{1}{k!} d^k f.$$

Lemma 5. *Let* $[f] \in \mathfrak{m}^2$. *Then*

$$d_0^2 f(v, w) = [\widetilde{v}, [\widetilde{w}, f]](0) = [\widetilde{w}, [\widetilde{v}, f]](0),$$

where $\widetilde{v}, \widetilde{w}$ *are any two germs of vector fields around zero extending* $v, w \in T_0\mathbb{R}^m$, *respectively.*

Proof. Note that

$$[\widetilde{v}, [\widetilde{w}, f]] - [\widetilde{w}, [\widetilde{v}, f]] = [[\widetilde{v}, \widetilde{w}], f](0) = d_0 f([\widetilde{v}, \widetilde{w}]) = 0$$

since $0 \in \mathrm{Crit}(f)$. Hence $[\widetilde{v}, [\widetilde{w}, f]](0) = [\widetilde{w}, [\widetilde{v}, f]](0)$. But the LHS can be expressed as

$$[\widetilde{v}, [\widetilde{w}, f]](0) = d([\widetilde{w}, f])(v),$$

which shows that it is independent of the choice of extension $\widetilde{v}$, whereas

$$[\widetilde{w}, [\widetilde{v}, f]](0) = d([\widetilde{v}, f])(w)$$

shows that it is independent of the extension $\widetilde{w}$. Now express f in coordinates and conclude that the quantity above equals $d_0^2 f(v, w)$ (**exercise**). ☐

Now recall if $B : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is a symmetric bilinear form, there exist integers $0 \leqslant \lambda, \nu \leqslant m$ and a linear basis $(e_i)_1^m$ of $\mathbb{R}^m$ with

$$B(e_i, e_j) = \begin{cases} -1 & \text{if } i = j \text{ and } i \leqslant \lambda, \\ +1 & \text{if } i = j \text{ and } \lambda < i \leqslant m - \nu, \\ 0 & \text{if } i \neq j \text{ or } i > m - \nu. \end{cases}$$

The integer ν is called the **nullity** of B; the form is called **non-degenerate** if $\nu = 0$. The integer λ, on the other hand, is called the **index** of B. Observe that $\nu, \lambda, (m - \lambda - \nu)$ are the dimensions of the maximal subspaces $W \subset \mathbb{R}^m$ where B restricts to zero, a negative-definite form, and a positive-definite form, respectively.

Lemma 6. *Let $[f] \in \mathfrak{m}^2$, and let $\mathrm{Jac}(f) \lhd \mathcal{E}$ denote the ideal spanned by the partial derivatives $\frac{\partial f}{\partial x_i}$. Then $d_0^2 f$ is non-degenerate only if $\mathrm{Jac}(f) = \mathfrak{m}$.*

Proof. Of course, $[f] \in \mathfrak{m}^2$ implies that $\mathrm{Jac}(f) \subset \mathfrak{m}$, so one inclusion always holds.

Suppose $d_0^2 f$ were non-singular. Then $d_0(df) = d_0^2 f : \mathbb{R}^m \to \mathrm{Hom}(\mathbb{R}^m, \mathbb{R})$ is a linear isomorphism; hence by the Inverse Function Theorem, we can express the coordinates x_i as

$$x_i = x_i(\partial f/\partial x_1, ..., \partial f/\partial x_m) \quad \Longrightarrow x_i = \sum_j a_{ij} \cdot \frac{\partial f}{\partial x_j},$$

for some $a_{ij} \in \mathcal{E}$. Since the x_i's span $\mathfrak{m}$, we have $\mathrm{Jac}(f) = \mathfrak{m}$. □

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Having described the local picture, we can transfer our definitions to the manifold setting :

Definition 8. *The* **Hessian** $\mathrm{Hess}_x(f)$ *is the bilinear form*

$$T_xM \times T_xM \to \mathbb{R}, \quad \mathrm{Hess}_x(f)(v, w) := d_x^2 f(v, w),$$

corresponding to the critical point $x \in \mathrm{Crit}(f)$. A critical point $x \in \mathrm{Crit}(f)$ is called **non-degenerate** *if $\mathrm{Hess}_x(f)$ is non-singular. If x is a non-degenerate critical point, its* **index** *$\lambda = \lambda(f, x)$ is*

$$\lambda(f, x) := \max\{\dim W : \mathrm{Hess}_x(f)|_W \text{ is negative-definite.}\}$$

A function $f \in C^\infty(M)$ will be called **Morse** *if all of its critical points are non-degenerate.*

We will write $\mathrm{Morse}(M) \subset C^\infty(M)$ for the subspace of Morse functions.

Lemma 7. *(1) $\mathrm{Morse}(M)$ is open in the strong C^2-topology, $\mathrm{Morse}(M) \subset C_S^2(M)$.*
(2) $\lambda(f, x) + \lambda(-f, x) = \dim M$.

Proof. Immediate. □

We wish to prove now :

Theorem 7 (Morse Lemma)**.** *If $f \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$, there exists an embedding*

$$\psi : 0 \in U \hookrightarrow \mathbb{R}^m, \quad \psi^* f = \frac{1}{2}\mathrm{Hess}_x(f) \in \mathcal{E},$$

where we regard $\mathrm{Hess}_x(f)$ as a smooth function $\mathrm{Hess}_x(f) : T_xM \to \mathbb{R}$ by the rule $v \mapsto \mathrm{Hess}_x(v, v)$.

We need a technical lemma first.

Lemma 8 (Auxiliary Lemma)**.** *If $f \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$, and $\delta \in \mathfrak{m}^2$, there exists a time-dependent vector field w_t around zero, $t \in [0, 1]$, for which $[w_t, f + t\delta] = -\delta$ and $w_t(0) = 0$ for all t.*

Proof. Note that $\delta \in \mathfrak{m}^3$ implies that $\nabla\delta \in \mathfrak{m}^2$, so

$$\nabla\delta = B(x)x, \quad B(0) = 0.$$

On the other hand, $\operatorname{Jac}(f) = \mathfrak{m}$, so $x = A(x)\nabla f$. Hence

$$\begin{cases} x = A(x)\left(\nabla(f + t\delta)\right) - tA(x)\nabla\delta \\ \nabla\delta = B(x)x \end{cases} \quad \Longrightarrow \quad (\mathrm{id} + tA(x)B(x))\, x = A(x)\nabla(f + t\delta).$$

Now, $B(0) = 0$ ensures that

$$x = C_t(x)\nabla(f + t\delta), \quad C_t(x) := (\mathrm{id} + tA(x)B(x))^{-1} A(x),$$

which means that each of germs of the coordinate functions x_i can be written as

$$x_i = [v_t^i, f + t\delta]$$

for some germ of time-dependent vector field v_t^i.

Now write $\delta = \sum a_{ij}x_ix_j$ and let

$$w_t := \sum_{i,j} a_{ij}x_jv_t^i;$$

then $[w_t, f + t\delta] = -\delta$ as promised, and $w_t(0) = 0$ for all t. □

Proof of Theorem 7. First observe that

$$f_t := (1 - t)f + \frac{t}{2}\operatorname{Hess}(f) = f + t\delta, \quad \delta := \frac{1}{2}\operatorname{Hess}(f) - f, \quad t \in [0, 1],$$

defines a smooth family $f_t \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$. Note that $\delta \in \mathfrak{m}^3$.

We seek a germ of isotopy ψ_t around 0, such that $\psi_t(0) = 0$ and

$$\psi_t^* f_t = f, \quad t \in [0, 1]$$

The latter condition is equivalent to

$$0 = \frac{d}{dt}(\psi_t^* f_t) \quad \Longleftrightarrow \quad L(w_t)f_t + \delta = 0,$$

and the former to $w_t(0) = 0$, where w_t denotes the germ of time-dependent vector field corresponding to ψ_t.

But by the Auxiliary Lemma 8, such w_t exists. □

Definition 9. *If $f \in C^\infty(M)$ and $p \in \operatorname{Crit}(f)$ is non-degenerate, a* **Morse chart** *around p is an embedding $\psi : U \hookrightarrow M$ of an open around $0 \in \mathbb{R}^m$ putting f in normal form :*

$$\psi^* f = Q_{\lambda(f,p)} + f(p),$$

where $Q_{\lambda(f,p)}$ stands for the standard quadratic form of index $\lambda = \lambda(f, p)$, $Q_{\lambda(f,p)} = -\sum_1^\lambda x_i^2 + \sum_{\lambda+1}^m x_i^2$.

2.2. Exercises.

(1) If $f \in \operatorname{Morse}(M)$ and $f' \in \operatorname{Morse}(M')$ $i = 0, 1$, then $F := \operatorname{pr}_M^* f + \operatorname{pr}_{M'}^* f' \in \operatorname{Morse}(M \times M')$. Determine the critical points of F and their indices in terms of those of f, f'.
(2) Give an example of isolated and non-isolated *degenerate* critical points.
(3) Show that if $[f] \in \mathfrak{m} \setminus \mathfrak{m}^2$, then f has the same behavior as $d_x f$.
(4) Show that if $f \in \operatorname{Morse}(M^m)$ and $|\operatorname{Crit}(f)| = 2$, then M is homeomorphic to $\mathbb{S}^m$.
(5) Show that every symmetric bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is equivalent to (exactly) one of the form $-\sum_1^\lambda x_i^2 + \sum_{\lambda+1}^n x_i^2$, $0 \leqslant \lambda \leqslant n$.

(6) Let $\mathfrak{F}$ be a family of closed subsets in a manifold M, with the property that, for every isotopy ϕ^t of M, we have

$$F \in \mathfrak{F} \quad \Longrightarrow \quad \phi^1(F) \in \mathfrak{F}.$$

(a) Let $f : M \to \mathbb{R}$ be any smooth function, and define

$$\operatorname{minmax}(f, \mathfrak{F}) := \inf_{F \in \mathfrak{F}} \sup_{x \in F} f(x).$$

Show that $\operatorname{minmax}(f, \mathfrak{F}) \in \operatorname{Crit}(f)$.

(b) Show that (a) remains true if we drop the assumption that M be compact, and have instead that M has a complete Riemannian metric, and f is bounded below and satisfies Palais' Condition C.[24].

(7) The **Ljusternik-Schnirrelman category** of a topological space X is the least number of *contractible*, closed sets needed to cover X :

$$\operatorname{cat}(X) := \inf \{|\mathfrak{U}| : \mathfrak{U} \text{ closed cover by contractibles}\};$$

if no finite such cover exists, $\operatorname{cat}(X) := \infty$.

Show that any smooth function $f : M \to \mathbb{R}$ on a compact M has at least $\operatorname{cat}(M)$ critical points.

(8) The **cuplength** of a topological space X is the largest number of non-trivial cup-products on X. That is, it is the largest k for which there exist :

- A ring A;
- Cohomology classes $\alpha_i \in H^{d_i}(X; A)$, $1 \leqslant i \leqslant k-1$,

with $\alpha_1 \cup \cdots \cup \alpha_{k-1} \neq 0 \in H^d(X; A)$, $d = \sum_1^{k-1} d_i$. Show that if X is a connected, $\operatorname{cuplength}(X)$ is bound above by $\operatorname{cat}(X)$.

3. Lecture Three. Abundance of Morse functions.

3.1. **Thom Transversality Theorem.** Recall that if M, M' are smooth manifolds, we say that $f, f' \in C^\infty(M, M')$ have the same **k-jet** at $x \in M$ iff all the partial derivatives of f and f' at x agree up to order k, in which case we write $j_k f(x) = j_k f'(x)$.

The collection

$$J_k(M, M') := \{j_k f(x) : f \in C^\infty(U, M'), x \in U\}$$

of all k-jets of (partially defined) maps $M \to M'$ has a natural structure of smooth manifold. It comes equipped with **source-** and **target** maps,

$$s : J_k(M, M') \to M, \quad j_k f(x) \mapsto x$$
$$t : J_k(M, M') \to M', \quad j_k f(x) \mapsto f(x);$$

which are fibre bundles, and bundle maps

$$p^k_{k-1} : J_k(M, M') \to J_{k-1}(M, M'), \quad j_k f(x) \mapsto j_{k-1} f(x)$$

so that we have commuting diagrams

$$\begin{tikzcd}
& J_k(M, M') \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[dd, "p^k_{k-1}"'] & \\
M & & M' \\
& J_{k-1}(M, M') \arrow[ul, "s"] \arrow[ur, "t"'] &
\end{tikzcd}$$

There is also an assignment

$$j_k : C^\infty(M, M') \to C^\infty(M, J_k(M, M')), \quad f \mapsto [x \mapsto j_k f(x)],$$

which we refer to as the **k-jet map**.

Recall that a subspace A of a topological space X is called **residual** if it is the countable intersection of open, dense subspaces :

$$A = \bigcap_{n \geqslant 0} U_n, \quad U_n \subset X \text{ open and } \operatorname{Cl} U_n = X, \quad \forall n.$$

A topological space X is called **Baire** if every residual subspace is dense.

Theorem 8. *A residual subspace of a complete metric space is dense. Every weakly closed subspace of $C^r_S(M, M')$ is a Baire space.*

Proof. See [10]. □

We can now remind the reader of :

Theorem 9 (Thom Transversality Theorem, v. 1)**.** *If $X \subset J_k(M, M')$ is a submanifold, then the space of $f \in C^r(M, M')$ with $j_k f \pitchfork X$ is residual in $C^r_S(M, M')$ for $r > k$, and is open if X is closed.*

Aside : Multijet bundles

We will make good use of an extension of Thom Transversality, whose setting we describe.

Fix an integer $l > 0$ and consider

$$J_k^{(l)}(M,M') \subset \prod_1^l J_k(M,M'), \quad J_k^{(l)}(M,M') := (\prod_1^k s)^{-1} M^{(l)},$$

where

$$M^l \supset M^{(l)} := \{(x_1,...,x_l) : i \neq j \Longrightarrow x_i \neq x_j\}.$$

Then clearly $J_k^{(l)}(M,M')$ is a bundle over $M^{(l)}$, with projection

$$s^{(l)} : (j_k f_1(x_1),...,j_k f_l(x_l)) \mapsto (x_1,...,x_l),$$

and there is an induced **multijet map**

$$j_k^{(l)} : C^k(M,M') \to C^0(M^{(l)}, J_k^{(l)}(M,M'))$$

$$j_k^{(l)} f : M^{(l)} \ni (x_1,...,x_l) \mapsto (j_k f(x_1),...,j_k f(x_l)) \in J_k^{(l)}(M,M').$$

Theorem 10 (Thom Transversality Theorem, v. 2)**.** *If $X \subset J_k^{(l)}(M,M')$ is a submanifold, then the space of $f \in C^r(M,M')$ with $j_k^{(l)} f \pitchfork X$ is residual in $C_S^r(M,M')$ for $r > k$, and is open if X is closed.*

Proof. See [7]. □

Now we put these ideas to use.

Definition 10. *The* **singularity set** $S_1 \subset J_1(M,\mathbb{R})$ *is the subspace defined by*

$$S_1 = \{j_1 f(x) : d_x f = 0\}.$$

Lemma 9. *S_1 is a closed submanifold, of codimension* $\operatorname{codim}(S_1 \subset J_1(M,\mathbb{R})) = \dim M$.

$x \in \operatorname{Crit}(f)$ iff $j_1 f(x) \in S_1$. Moreover, x is non-degenerate iff $j_1 f \pitchfork S_1$ at x.

Corollary 3. $\operatorname{Morse}(M) \subset C_S^2(M,\mathbb{R})$ *is open and dense. If $f \in \operatorname{Morse}(M)$, $\operatorname{Crit}(f)$ is discrete.*

Proof. Combine Lemma 9 with Theorem 9 for the first statement. For the second, observe that $\operatorname{codim}(\operatorname{Crit}(f) \subset M) = \dim M$, so $\operatorname{Crit}(f)$ is a zero-dimensional submanifold. □

Definition 11. *A Morse function $f \in \operatorname{Morse}(M)$ is called* **resonant** *if there exist distinct critical points $x, y \in \operatorname{Crit}(f)$ at the same critical value : $f(x) = f(y)$. Otherwise it is called* **non-resonant***, and the space of all such will be written* $\operatorname{Morse}_{\neq}(M)$.

Lemma 10. $\operatorname{Morse}_{\neq}(M) \subset C_S^2(M,\mathbb{R})$ *is open and dense.*

Proof. First observe that $\operatorname{Morse}_{\neq}(M) \subset C_S^2(M,\mathbb{R})$ is clearly open, so we need only show that it contains a dense subspace.

Consider the multijet bundle $J_1^{(2)}(M,\mathbb{R}) \to M^{(2)} = M \times M \setminus \Delta_M$, and let $S_{\neq} \subset J_1^{(2)}(M,\mathbb{R})$ be the subspace defined by

$$S_{\neq} := (S_1 \times S_1) \cap (t \times t)^{-1}(\Delta_{\mathbb{R}}).$$

One readily sees that $S_{\neq}$ is a submanifold of codimension $2\dim M + 1$, and hence $j_1^{(2)} f \pitchfork S_{\neq}$ at (x_1, x_2) means $j_1^{(2)} f(x_1,x_2) \notin S_{\neq}$.

By Theorem 10, the subspace $U \subset C_S^2(M,\mathbb{R})$ of those f with $j_1^{(2)} f \pitchfork S_{\neq}$ is open and dense; thus $\operatorname{Morse}(M) \cap U$ is open and dense. But $j_1^{(2)} f$ maps (x_1, x_2) into $S_{\neq}$ iff $d_{x_1} f = 0 = d_{x_2} f$ and $f(x_1) = f(x_2)$, so $\operatorname{Morse}_{\neq}(M) \supset \operatorname{Morse}(M) \cap U$. □

3.2. **Concatenating and Factorizing Cobordisms.**

In view of Lemma 10, any $f \in C^\infty(M)$ can be perturbed ever so slightly to a non-resonant Morse function.

Suppose M is compact, so that $\mathrm{Crit}(f)$ is *finite*. Order the critical values

$$\{c_1 < c_2 < \cdot < c_N\} = f\,\mathrm{Crit}(f),$$

and let $-\infty = a_0 < a_1 < \cdots a_{N-1} < a_N = +\infty$, with $c_i \in (a_{i-1}, a_i)$ for every $1 \leqslant i \leqslant N$.

Then

$$\mathcal{C}_i := \left(W_i, f^{-1}(a_{i-1}), f^{-1}(a_i)\right), \quad W_i := f^{-1}[a_{i-1}, a_i].$$

are cobordisms, and $M = \cup_i W_i$. Note that $f \pitchfork a_i$ for every $0 < i < N$, and $f_i := f|_{W_i}$ contains a single critical point. We give this situation a special name :

Definition 12. *A cobordism $\mathcal{C} = (W; M_0, M_1)$ is called* **elementary** *if there exists a smooth function $f : W \to [a,b]$, with $f \pitchfork \partial[a,b]$, $f^{-1}(a) = \partial_0 W$, $f^{-1}(b) = \partial_1 W$, and $\mathrm{Crit}(f) = \{p\}$, with $a < f(p) < b$.*

Definition 13. *Let W be a manifold with boundary $\partial W \hookrightarrow W$. By a* **distinguished submanifold** *$X \subset W$ we will refer to either a connected component of the boundary $X \subset \partial W$, or to a cooriented interior submanifold $X \subset (W \smallsetminus \partial W)$.*

A **collar** *of a distinguished submanifold X is an embedding $c : X \times I(X,\varepsilon) \hookrightarrow W$ with $c|_X = \mathrm{id}_X$, and $c_*(\partial/\partial t)$ pointing inwards if $X \subset \partial W$, and in the positive coorientation if $X \subset (W \smallsetminus \partial W)$; here $I(X,\varepsilon) = (-\varepsilon,\varepsilon)$ if X is interior and $I(X,\varepsilon) = [0,\varepsilon)$ if X lies in the boundary.*

Lemma 11 (Collars)**.**

(1) Collars exist.

(2) If $c, c' : X \times I(X,\varepsilon) \hookrightarrow W$ are collars, there is $0 < \delta \leqslant \varepsilon$ and a homotopy of collars $C : X \times I(X,\delta) \times [0,1] \to W$ joining $c|_{X\times I(X,\delta)}$ to $c'|_{X\times I(X,\delta)}$.

(3) If $C : X \times I(X,\delta) \times [0,1] \to W$ is a homotopy of collars, there is a collar $\overline{c} : X \times I(X,\delta) \hookrightarrow W$ with

$$\overline{c}|_{X\times I(X,\delta/3)} = C_1|_{X\times I(X,\delta/3)}$$
$$\overline{c}|_{X\times (I(X,\delta)\smallsetminus I(X,2\delta/3))} = C_0|_{X\times (I(X,\delta)\smallsetminus I(X,2\delta/3))}$$

Proof. (1) Using a partition of unity, one constructs on an open $U \subset W$ containing X a vector field $w \in \mathfrak{X}(U)$ with w pointing inwards if $X \subset \partial W$, and w in the positive coorientation if X is interior.

Let $\phi : U \times \mathbb{R} \supset \mathrm{dom}(\phi) \to U$ denote the local flow of w, and choose any embedding $\psi : X \times I(X,\varepsilon) \hookrightarrow \mathrm{dom}(\phi)$ with $\psi|_{X\times\{0\}}$ the inclusion $X \hookrightarrow \mathrm{dom}(\phi)$. Then $c := \phi \circ \psi$ is a collar.

(2) Let $v := c_*(\partial/\partial t), v' := c'_*(\partial/\partial t)$ be defined in a common open $X \subset U$. Define $v_s := (1-s)v + sv' \in \mathfrak{X}(U)$, for $s \in [0,1]$, and let

$$\phi_{v_s} : U \times \mathbb{R} \supset \mathrm{dom}(\phi_{v_s}) \longrightarrow U$$

denote the local flow of v_s. Choose a homotopy of embeddings $\psi_s : X \times I(X,\varepsilon) \hookrightarrow \mathrm{dom}(\phi_{v_s})$, $0 \leqslant s \leqslant 1$, with $\psi_s|_X$ the inclusion, and set $C_s := \phi_{v_s} \circ \psi_s : X \times I(X,\varepsilon) \hookrightarrow W$.

(3) Let $s \mapsto w_s$ denote the time-dependent vector field $\frac{dC_s}{ds} \in \mathfrak{X}(\mathrm{im}\, C_s)$, and note that $w_s(x) = 0$ for all $x \in X$ and $s \in [0,1]$; hence w_s has bounded velocity on some $U'_s \supset X$. Choose then a smooth function $\varrho : X \times I(X,\varepsilon) \times [0,1] \to \mathbb{R}$, with $\varrho_s = 1$ on a smaller open $U''_s \subset U'_s$ around X, and set $\overline{w}_s := \varrho_s w_s \in \mathfrak{X}(W)$. Then $\overline{w}$ has bounded velocity, and thus generates an isotopy ϕ^s of W with $d_x\phi_s = \mathrm{id}$ for all $x \in X$ and $s \in [0,1]$, and $\phi^1 C_0$ agrees with C_0 away from X, and with C_1 around it.

□

Corollary 4. *Suppose W, W' are smooth manifolds with boundary, that $X \subset \partial W$ be a sum of* outgoing *connected, and that $h : X \hookrightarrow \partial W'$ embeds X as a sum of* incoming *connected compopnents of $\partial W'$. Then the topological space $W \cup_h W'$ carries a canonical structure of smooth manifold with boundary, and*

$$\partial(W \cup_h W') = (\partial W \diagdown X) \coprod (\partial W' \diagdown h(X))$$

Proof. Suppose for simplicity that X is connected; the general case is argued component-by-component.

We need first introduce a smooth structure on $W \cup_h W'$. Choose collars

$$c : X \times (-\varepsilon, 0] \hookrightarrow W, \quad c' : h(X) \times [0, \varepsilon) \hookrightarrow W'$$

and define the space $W \cup_{h,c} W'$ according to the diagram

$$\begin{tikzcd} X \times ((-\varepsilon,\varepsilon)\diagdown 0) \arrow[r, "H"] \arrow[d] & (W\diagdown X)\coprod(W'\diagdown h(X)) \arrow[d, dashed] \\ X \times (-\varepsilon,\varepsilon) \arrow[r, dashed] & W \cup_{h,c} W' \end{tikzcd}$$

where

$$H(x,t) = \begin{cases} c(x,t) & \text{if } t < 0; \\ c'(h(x),t) & \text{if } t > 0. \end{cases}$$

This exhibits $W \cup_{h,c} W'$ as a *smooth* manifold with the boundary as in the statement.

We need now show that the recipe above is independent of the choices of collars c, c' up to a diffeomorphism.

So suppose γ, γ' are two different choices of collars, and let $W \cup_{h,\gamma} W'$ denote the manifold arising from those choices. Then note that the identity maps $\mathrm{id}_W, \mathrm{id}_{W'}$, glue to a homeomorphism

$$G : W \cup_{h,c} W' \longrightarrow W \cup_{h,\gamma} W'.$$

On the $X \times (-\varepsilon, +\varepsilon)$ part of those manifolds, G reads

$$G = \begin{cases} \gamma c^{-1} & \text{on } \operatorname{im} c; \\ \gamma' c'^{-1} & \text{on } \operatorname{im} c'. \end{cases}$$

According to Lemma 11, c, c' can be modified to a collars $\overline{c}, \overline{c'}$, with

$$\overline{c} = \begin{cases} c & \text{on } X \times (-\varepsilon/3, 0]; \\ \gamma & \text{on } X \times (-\varepsilon, -2\varepsilon/3). \end{cases}, \quad \overline{c'} = \begin{cases} c' & \text{on } h(X) \times [0, \varepsilon/3); \\ \gamma' & \text{on } h(X) \times (2\varepsilon/3, \varepsilon). \end{cases}$$

We then modify G to a *diffeomorphism* $\overline{G} : W \cup_{h,c} W' \xrightarrow{\sim} W \cup_{h,\gamma} W'$,

$$\overline{G} := \begin{cases} G & \text{outside } X \times (-\varepsilon, \varepsilon); \\ \gamma \overline{c}^{-1} & \text{on } \operatorname{im} c; \\ \gamma' \overline{c'}^{-1} & \text{on } \operatorname{im} c'. \end{cases}.$$

□

Definition 14. *We refer to $W \cup_h W'$ as the* **concatenation** *of W, W' along h.*

Example 6. *Let W be any manifold with boundary, and define $2W := W \cup_{\mathrm{id}_{\partial W}} \overline{W}$, the* **double** *of W. Note that $\partial(2W) = \varnothing$.*

Note that, by its very construction, concatenation is 'distributive', in the sense that if we are given a further manifold with boundary W'', $Y \subset \partial W'$ is outgoing, and $h' : Y \hookrightarrow \partial W''$ is an incoming embedding, then there is a natural identification

$$(W \cup_h W') \cup_{h'} W'' \simeq W \cup_h (W' \cup_{h'} W'')\,.$$

Definition 15. *A* **factorization** *of a manifold with boundary W is a presentation as a concatenation of manifolds with boundary :*

$$W = W_0 \cup_{h_1} W_1 \cup_{h_2} \cdots \cup_{h_k} W_k.$$

Lemma 12. *Every cobordism $\mathcal{C}$ can be factorized into elementary cobordisms.*

Proof. Let $\mathcal{C} = (W; M_0, M_1)$ be a cobordism. Double W to the manifold (without boundary) $2W$, and note that ∂W embeds as a compact submanifold of $2W$.

Choose any $f' : 2W \to [-1, +1]$ with $f' \pitchfork \partial\mathbb{D}^1$ and $f'^{-1}\partial\mathbb{D}^1 = \partial W$. Use Lemma 10 to perturb f' to $f'' \in \mathrm{Morse}_{\neq}(2W)$; choose f'' so C^1-close to f' so that $\partial\mathbb{D}^1$ are regular values for $f'_t := (1-t)f' + tf''$, $0 \leqslant t \leqslant 1$. Then there is a homotopy of embeddings $\psi : \partial W \times [0,1] \to 2W$ tracking $f_t'^{-1}\partial\mathbb{D}^1$:

$$f'_t \psi_t(\partial_i W) = i, \quad i = 0, 1.$$

By the Isotopy Extension Lemma 13 below, ψ can be extended to an isotopy $\varphi : 2W \times [0,1] \to 2W$; then

$$f := f'' \circ \varphi_1|_{2W \smallsetminus (\overline{W} \smallsetminus \overline{W})} \in C^\infty(W)$$

is transverse to $\partial\mathbb{D}^1$ and pulls it back to ∂W, and is a non-resonant Morse function in the interior of W. Now choose $a_i \in \mathbb{R} \smallsetminus f\,\mathrm{Crit}(f)$ such that every $c \in f\,\mathrm{Crit}(f)$ lies in exactly one interval (a_i, a_{i+1}); then the concatenation of the cobordisms

$$\mathcal{C}_i := \big(W_i, f^{-1}(a_{i-1}), f^{-1}(a_i)\big)$$

is diffeomorphic to W. □

Lemma 13 (Isotopy Extension Lemma)**.** *Let W be a manifold with boundary, and $X \subset W$ a closed submanifold, with either $X \subset (W \smallsetminus \partial W)$ or $X \subset \partial W$. Then every homotopy of embeddings $\psi : X \times [0,1] \to W$, $\psi_t : X \hookrightarrow W$, whose velocity $\frac{d\psi_t}{dt}$ is bounded, extends to an isotopy $\varphi : W \times [0,1] \to W$.*

Proof. **Case 1 : $X \subset (W \smallsetminus \partial W)$.**

Consider

$$\widehat{\psi} : X \times [0,1] \longrightarrow W \times [0,1], \quad \widehat{\psi}(x,t) = (\psi_t(x), t).$$

The hypotheses ensure that $\widehat{\psi}$ is a closed embedding, and that

$$\widehat{w} := \frac{d\psi_t}{dt} + \partial/\partial t$$

is defined along its image and has bounded velocity.

Choose :

- a tubular neighborhood
$$(W \smallsetminus \partial W) \times I \supset E \xrightarrow{p} \widehat{\psi}(X \times [0,1]);$$
- a smooth function $\varrho \in C^\infty(E)$, with $\varrho = 1$ around $\widehat{\psi}(X \times [0,1])$, and whose support meets every fibre of p in a compact set;
- an Ehresmann connection $\mathrm{hor} : \mathfrak{X}(\widehat{\psi}(X \times [0,1])) \longrightarrow \mathfrak{X}(E)$.

Then set $w := \varrho\,\mathrm{hor}(\widehat{w}) \in \mathfrak{X}(W \times [0,1])$ and observe that $w = w_t + f\partial/\partial t$, where $w_t \in \mathfrak{X}(W)$ is supported in the interior of W, and extends $\frac{d\psi_t}{dt}$. Hence w_t gives rise to an isotopy of W extending ψ.

Case 2 : $X \subset \partial W$.

Apply Case 1 twice, first to $X \subset \partial W$, and then to $\partial W \subset W$. □

3.3. **Exercises.**

(1) Show that $J_k(M, M')$ is indeed a smooth manifold, and compute its dimension.

(2) Show that $j_k : C^k(M, M') \to C^0(M, J_k(M, M'))$ is continuous in both the weak and the strong topologies, and has closed image in the weak topology.

(3) Let $M \subset \mathbb{R}^N$ be a submanifold. For each $y \in \mathbb{R}^N$, let $f_y : M \to \mathbb{R}$ denote $x \mapsto \|y - x\|^2$. Show that for y *generic*, $f_y \in \mathrm{Morse}(M)$.(Meaning that the set of points for which the stated property holds is residual).

(4) Compute $\pi_n(\mathbb{S}^m)$ for all $m > n \geqslant 0$.

(5) Two compact manifolds M_0^m, M_1^m are called **(oriented) cobordant** if there exists a (oriented) cobordism $\mathcal{C} = (W; M_0, M_1)$. Show that :
  (a) Being (oriented) cobordant to is an equivalence relation.
  (b) The sets $\mathcal{N}_m$, Ω_m of equivalence classes under cobordism and oriented cobordism relations, respectively, are *abelian groups* under disjoint union $\coprod$.
  (c) If $f, f' : M \to M'$ are homotopic, and transverse to a closed submanifold $X \subset M'$, then $f^{-1}X$ and $f'^{-1}X$ are cobordant. If M, M' and X are orientable, $f^{-1}X$ and $f'^{-1}X$ are oriented cobordant.
  (d) Compute $\mathcal{N}_i$ and Ω_i, for $i = 0, 1$.

(6) Let M, M' be compact smooth manifolds, and let $G := \mathrm{Diff}(M') \times \mathrm{Diff}(M)$ act on $C^\infty(M, M')$ by

$$(\psi, \varphi) : f \mapsto \psi \circ f \circ \varphi^{-1},$$

where G is endowed with the C^∞ topology. A map f is called **stable** if every f' close enough to f lies in the same orbit as f.

Show that $f \in C^\infty(M, \mathbb{R})$ is stable only if $f \in \mathrm{Morse}_{\neq}(M)$.[5]

[5]We will see later that $\mathrm{Morse}_{\neq}(M)$ is precisely the space of stable functions on M.

4. Lecture Four. Passing a critical level set

4.1. Surgery. For every $1 \leqslant \lambda < m$, consider the "standard" diffeomorphisms

$$\operatorname{std}_\lambda : \mathbb{S}^{\lambda-1} \times (\mathring{\mathbb{D}}^{m-\lambda+1} \setminus 0) \xrightarrow{\simeq} (\mathring{\mathbb{D}}^{\lambda} \setminus 0) \times \mathbb{S}^{m-\lambda}$$
$$\operatorname{std}_\lambda : (u, \theta v) \mapsto (\theta u, v), \quad (u,v) \in \mathbb{S}^{\lambda-1} \times \mathbb{S}^{m-\lambda}, \quad \theta \in (0,1).$$

Fix an embedding

$$\varphi : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \hookrightarrow M^m,$$

and consider the *smooth* manifold $\operatorname{Surg}(M,\varphi)$ defined by the pushout diagram

$$\begin{tikzcd} \mathbb{S}^{\lambda-1} \times (\mathring{\mathbb{D}}^{m-\lambda+1} \setminus 0) \arrow[r, "\varphi"] \arrow[d, "\operatorname{std}_\lambda"'] & M \setminus \varphi(\mathbb{S}^{\lambda-1}) \arrow[d, dashed] \\ \mathring{\mathbb{D}}^{\lambda} \times \mathbb{S}^{m-\lambda} \arrow[r, dashed] & \operatorname{Surg}(M,\varphi). \end{tikzcd}$$

Observe that $\operatorname{Surg}(M,\varphi)$ comes equipped with a canonical embedding $\operatorname{Surg}(\varphi) : \mathring{\mathbb{D}}^{\lambda} \times \mathbb{S}^{m-\lambda} \hookrightarrow \operatorname{Surg}(M,\varphi)$. Producing $\operatorname{Surg}(M,\varphi)$ out of M has the effect of removing a $(\lambda-1)$-sphere, embedded with trivial normal bundle in M, and replacing it by a $(m-\lambda)$-sphere, also embedded with trivial normal bundle.

Definition 16. *We say that* $\operatorname{Surg}(M,\varphi)$ *is obtained from* M *by a* **surgery of type** λ.

Lemma 14. *If* $\varphi_t : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \hookrightarrow M$ *is a homotopy of embeddings, then* $\operatorname{Surg}(M,\varphi_0) \simeq \operatorname{Surg}(M,\varphi_1)$.

Proof. Extend $\frac{d\varphi_t}{dt} \in \mathfrak{X}(\operatorname{im}\varphi_t)$ to a globally defined (time-dependent) vector field $w_t \in \mathfrak{X}(M)$. We can further demand that the support of w_t be a small neighborhood of $\operatorname{im}\varphi_t$. Denote by ϕ^t the isotopy it generates, and observe that

$$\phi^t(\varphi_t(u,\theta v)) = \varphi_t(u,\theta v).$$

Then

$$\phi^1 \coprod \operatorname{id} : (M \setminus \varphi_0(\mathbb{S}^{\lambda-1}) \coprod \mathring{\mathbb{D}}^{\lambda} \times \mathbb{S}^{m-\lambda} \xrightarrow{\simeq} (M \setminus \varphi_1(\mathbb{S}^{\lambda-1}) \coprod \mathring{\mathbb{D}}^{\lambda} \times \mathbb{S}^{m-\lambda}$$

descends to a diffeomorphism $\operatorname{Surg}(M,\varphi_0) \xrightarrow{\simeq} \operatorname{Surg}(M,\varphi_1)$. $\square$

4.2. A closer look at model singularities. Let $L_\lambda \subset \mathbb{R}^\lambda \times R^{m-\lambda+1}$ be the subspace defined by

$$L_\lambda := \{(x,y) : -1 \leqslant Q_\lambda(x,y) \leqslant +1, |x||y| < \sinh 1 \cosh 1\},$$

where as usual Q_λ denotes $Q_\lambda(x,y) = -|x|^2 + |y|^2$.

Note that L_λ is a smooth manifold with boundary $\partial L_\lambda = \partial_{\text{left}} L_\lambda \coprod \partial_{\text{right}} L_\lambda$, where

$$\partial_{\text{left}} L_\lambda := \{(x,y) \in L_\lambda : Q_\lambda(x,y) = -1\}$$
$$\partial_{\text{right}} L_\lambda := \{(x,y) \in L_\lambda : Q_\lambda(x,y) = +1\}.$$

We let $\mathbb{R}_\times$ denote

$$\mathbb{R}_\times := (\mathbb{R}^\lambda \setminus 0) \times (\mathbb{R}^{m-\lambda+1} \setminus 0) \subset \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda+1}.$$

Lemma 15. *There exist diffeomorphisms*

$$\varphi_{left} : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \xrightarrow{\simeq} \partial_{left} L_\lambda$$
$$\varphi_{right} : \mathring{\mathbb{D}}^{\lambda} \times \mathbb{S}^{m-\lambda} \xrightarrow{\simeq} \partial_{right} L_\lambda$$
$$\mathrm{std}^\lambda : \partial_{left} L_\lambda \cap \mathbb{R}_\times \xrightarrow{\simeq} \partial_{right} L_\lambda \cap \mathbb{R}_\times,$$

such that

$$\begin{tikzcd}
\partial_{left} L_\lambda \cap \mathbb{R}_\times \arrow[r, "\varphi\varphi_{left}^{-1}"] \arrow[d, "\mathrm{std}^\lambda"'] & M \setminus \varphi(\mathbb{S}^{\lambda-1}) \arrow[d, dashed] \\
\partial_{right} L_\lambda \arrow[r, dashed] & \mathrm{Surg}(M, \varphi)
\end{tikzcd}$$

$$\begin{tikzcd}
\partial_{right} L_\lambda \cap \mathbb{R}_\times \arrow[r, "\mathrm{Surg}(\varphi)\varphi_{right}^{-1}"] \arrow[d, "\mathrm{std}^\lambda"'] & \mathrm{Surg}(M, \varphi) \setminus \mathrm{Surg}(\varphi)(\mathbb{S}^{m-\lambda}) \arrow[d, dashed] \\
\partial_{left} L_\lambda \arrow[r, dashed] & M
\end{tikzcd}$$

Proof. Define $\mathrm{std}^\lambda : \mathbb{R}_\times \xrightarrow{\simeq} \mathbb{R}_\times$ by the formula

$$\mathrm{std}^\lambda : (x, y) \mapsto \left(\frac{|x|}{|y|} x, \frac{|y|}{|x|} y \right),$$

and observe that std^λ is an *involution*, $\mathrm{std}^\lambda = (\mathrm{std}^\lambda)^{-1}$. Moreover, it induces a diffeomorphism

$$\partial_{\mathrm{left}} L_\lambda \cap \mathbb{R}_\times \xrightarrow{\simeq} \partial_{\mathrm{right}} L_\lambda \cap \mathbb{R}_\times,$$

which we still denote by std^λ.

Now define the diffeomorphisms

$$\varphi_{\mathrm{left}} : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \xrightarrow{\simeq} \partial_{\mathrm{left}} L_\lambda, \quad \varphi_{\mathrm{left}}(u, \theta v) = (u \cosh\theta, v \sinh\theta)$$
$$\varphi_{\mathrm{right}} : \mathring{\mathbb{D}}^{\lambda} \times \mathbb{S}^{m-\lambda} \xrightarrow{\simeq} \partial_{\mathrm{right}} L_\lambda, \quad \varphi_{\mathrm{right}}(\theta u, v) = (u \sinh\theta, v \cosh\theta).$$

Then

$$\begin{tikzcd}
\mathbb{S}^{\lambda-1} \times (\mathring{\mathbb{D}}^{m-\lambda+1} \setminus 0) \arrow[r, "\varphi_{\mathrm{left}}", "\simeq"'] \arrow[d, "\mathrm{std}_\lambda"', "\simeq"] & \partial_{\mathrm{left}} L_\lambda \cap \mathbb{R}_\times \arrow[d, "\simeq"', "\mathrm{std}^\lambda"] \\
(\mathring{\mathbb{D}}^{\lambda} \setminus 0) \times \mathbb{S}^{m-\lambda} \arrow[r, "\simeq", "\varphi_{\mathrm{right}}"'] & \partial_{\mathrm{right}} L_\lambda \cap \mathbb{R}_\times
\end{tikzcd}$$

commutes. Hence

$$\begin{tikzcd}
\partial_{\mathrm{left}} L_\lambda \cap \mathbb{R}_\times \arrow[r, "\varphi\varphi_{\mathrm{left}}^{-1}"] \arrow[d, "\mathrm{std}^\lambda"'] & M \setminus \varphi(\mathbb{S}^{\lambda-1}) \arrow[d, dashed] \\
\partial_{\mathrm{right}} L_\lambda \arrow[r, dashed] & \mathrm{Surg}(M, \varphi).
\end{tikzcd}$$

is also a pushout diagram. On the other hand, the pushout of the outer diagram in

$$\begin{tikzcd}
\mathbb{S}^{\lambda-1} \times (\mathring{\mathbb{D}}^{m-\lambda+1} \setminus 0) \arrow[r, "\varphi_{\mathrm{left}}"] \arrow[d, "\mathrm{std}_\lambda"'] & \partial_{\mathrm{left}} L_\lambda \cap \mathbb{R}_\times \arrow[r, "\varphi\varphi_{\mathrm{left}}^{-1}"] \arrow[d, "\mathrm{std}^\lambda"'] & M \setminus \varphi(\mathbb{S}^{\lambda-1}) \arrow[d] \\
(\mathring{\mathbb{D}}^{\lambda} \setminus 0) \times \mathbb{S}^{m-\lambda} \arrow[r, "\varphi_{\mathrm{right}}"] \arrow[d, "\mathrm{std}_\lambda"'] & \partial_{\mathrm{right}} L_\lambda \cap \mathbb{R}_\times \arrow[r, "\mathrm{Surg}(\varphi)\varphi_{\mathrm{right}}^{-1}"] \arrow[d, "\mathrm{std}^\lambda"'] & \mathrm{Surg}(M,\varphi) \setminus \mathrm{Surg}(\varphi)(\mathbb{S}^{m-\lambda}) \\
\mathbb{S}^{\lambda-1} \times (\mathring{\mathbb{D}}^{m-\lambda+1} \setminus 0) \arrow[r, "\varphi_{\mathrm{right}}"'] & \partial_{\mathrm{left}} L_\lambda \cap \mathbb{R}_\times &
\end{tikzcd}$$

is clearly $M \setminus \varphi(\mathbb{S}^{\lambda-1})$, as the top horizontal arrow equals φ and the left vertical one is identical. Hence

$$\begin{tikzcd}
\partial_{\mathrm{right}} L_\lambda \cap \mathbb{R}_\times \arrow[r, "\mathrm{Surg}(\varphi)\varphi_{\mathrm{right}}^{-1}"] \arrow[d, "\mathrm{std}^\lambda"'] & \mathrm{Surg}(M,\varphi) \setminus \mathrm{Surg}(\varphi)(\mathbb{S}^{m-\lambda}) \arrow[d, dashed] \\
\partial_{\mathrm{left}} L_\lambda \arrow[r, dashed] & M
\end{tikzcd}$$

□

Theorem 11. *There is an elementary cobordism* $(\mathcal{C}, f)$ *of index* λ *between* M *and* $\mathrm{Surg}(M, \varphi)$.

Proof. For every $(x, y) \in L_\lambda$, the curve

$$t \mapsto (tx, t^{-1}y), t > 0,$$

is orthogonal to the level sets $Q_\lambda = c$, $c \neq 0$.

Observe that

$$t = t(x,y) := \sqrt{\frac{1 + \sqrt{1 + 4|x|^2|y|^2}}{2|x|^2}} \Longrightarrow Q_\lambda(tx, t^{-1}y) = -1;$$

hence we obtain a diffeomorphism

$$\psi : L_\lambda \cap \mathbb{R}_\times \xrightarrow{\simeq} (\partial_{\mathrm{left}} L_\lambda \cap \mathbb{R}_\times) \times [-1, +1],$$
$$\psi : (x, y) \mapsto \big((t(x,y)x, t(x,y)^{-1}y), Q_\lambda(x,y)\big).$$

We can thus form the smooth manifold W by

$$\begin{tikzcd}
L_\lambda \cap \mathbb{R}_\times \arrow[r, "(\varphi\varphi_{\mathrm{left}}^{-1} \times \mathrm{id})\psi"] \arrow[d] & (M \setminus \varphi(\mathbb{S}^{\lambda-1}) \times [-1, +1] \arrow[d, dashed] \\
L_\lambda \arrow[r, dashed] & W
\end{tikzcd}$$

and note that

$$\partial W = \partial_0 W \coprod \partial_1 W,$$

where

$$\begin{tikzcd}
\partial_{\mathrm{left}} L_\lambda \cap \mathbb{R}_\times \arrow[r] \arrow[d] & (M \setminus \varphi(\mathbb{S}^{\lambda-1}) \arrow[d, dashed] \\
\partial_{\mathrm{left}} L_\lambda \arrow[r, dashed] & \partial_0 W
\end{tikzcd}$$

and

$$\begin{tikzcd}
\partial_{\mathrm{right}} L_\lambda \cap \mathbb{R}_\times \arrow[r] \arrow[d] & (M \setminus \varphi(\mathbb{S}^{\lambda-1}) \arrow[d, dashed] \\
\partial_{\mathrm{right}} L_\lambda \arrow[r, dashed] & \partial_1 W
\end{tikzcd}$$

so $\partial_0 W \simeq M$ and $\partial_1 W \simeq \mathrm{Surg}(M, \varphi)$.

Hence W is a cobordism between M and $\mathrm{Surg}(M, \varphi)$; to finish we must indicate the pertinent elementary Morse function $f \in \mathrm{Morse}(W)$. But observe that under the above identifications, the smooth map

$$\widetilde{f} : (M \setminus \varphi(\mathbb{S}^{\lambda-1}) \times [-1, +1] \coprod L_\lambda \longrightarrow \mathbb{R}$$
$$\widetilde{f}|_{(M \setminus \varphi(\mathbb{S}^{\lambda-1}) \times [-1,+1]} = \mathrm{pr}_2, \quad \widetilde{f}|_{L_\lambda} = Q_\lambda$$

descends to a smooth $f \in C^\infty(W)$ with the required properties.

□

On the other hand, suppose $(\mathcal{C}, f)$ is an elementary cobordism, where $f : W \to \mathbb{D}^1$ is an elementary Morse function with a unique critical point p of index λ at the level set 0.

We wish to define an embedding

$$\varphi : \mathbb{S}^{\lambda-1} \times \overset{\circ}{\mathbb{D}}{}^{m-\lambda+1} \hookrightarrow \partial_0 W$$

Fix a Morse chart $e : B_{2\varepsilon}^{m+1} \hookrightarrow W^{m+1}$ centred at p,

$$e(0) = p \in \mathrm{Crit}(f), \quad e^* f = Q_\lambda.$$

Then

$$\varphi' : \mathbb{S}^{\lambda-1} \times \overset{\circ}{\mathbb{D}}{}^{m-\lambda+1} \hookrightarrow f^{-1}(-\varepsilon),$$
$$(u, \theta v) \mapsto e(\sqrt{\varepsilon} u \cosh \theta, \sqrt{\varepsilon} u \sinh \theta)$$

embeds $\mathbb{S}^{\lambda-1} \times \overset{\circ}{\mathbb{D}}{}^{m-\lambda+1}$ in the *regular* level set $f = -\varepsilon$. The (local) flow ϕ^t of the vector field $w := -\frac{\nabla f}{\|\nabla f\|^2} \in \mathfrak{X}(M \setminus p)$, $\phi : (M \setminus p) \times \mathbb{R} \supset \mathrm{dom}(\phi) \longrightarrow M \setminus p$, determines a homotopy of embeddings

$$\varphi'_t : \mathbb{S}^{\lambda-1} \times \overset{\circ}{\mathbb{D}}{}^{m-\lambda+1} \hookrightarrow W,$$
$$\mathbb{S}^{\lambda-1} \times \overset{\circ}{\mathbb{D}}{}^{m-\lambda+1} \times [0, 1 - \sqrt{\varepsilon}] \to W, \quad ((u, \theta v), t) \mapsto \phi^{-t}(\varphi'(u, \theta v)),$$

and we set

$$\varphi := \varphi'_{1-\sqrt{\varepsilon}} : \mathbb{S}^{\lambda-1} \times \overset{\circ}{\mathbb{D}}{}^{m-\lambda+1} \hookrightarrow \partial_0 W$$

Observe that the choice of $\varepsilon > 0$ is immaterial, since the embeddings determined by any two choices according to the recipe above must coincide.

By the same token, we can drag the embedding

$$\Phi' : \mathbb{S}^{\lambda-1} \times \overset{\circ}{\mathbb{D}}{}^{m-\lambda+1} \hookrightarrow f^{-1}(\varepsilon), \quad (u, \theta v) \mapsto e(\sqrt{\varepsilon} u \sinh \theta, \sqrt{\varepsilon} u \cosh \theta)$$

along the flow of w from time $t = 0$ to $t = 1 - \sqrt{\varepsilon}$ to obtain an embedding

$$\Phi : \mathbb{S}^{\lambda-1} \times \overset{\circ}{\mathbb{D}}{}^{m-\lambda+1} \hookrightarrow \partial_1 W.$$

Definition 17. *We call the embeddings φ, Φ* **characteristic-** *and* **cocharacteristic embeddings** *of $(\mathcal{C}, f)$.*

Remark 3. *Note that the (co-)characteristic embedding depends on the choice of Morse chart, and also on the vector field ∇f which we used to drag objects around. Such choices will be implicit whenever we speak of such embeddings.*

Theorem 12. *If $(\mathcal{C}, f)$ is elementary of index λ, then $\partial_1 W \simeq \mathrm{Surg}(\partial_0 W, \varphi)$, for some characteristic embedding $\varphi : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \hookrightarrow \partial_0 W$.*

Proof. In terms of the notation above, one argues as in Theorem 11 to deduce that $f^{-1}(\varepsilon) \simeq \mathrm{Surg}(f^{-1}(-\varepsilon), \varphi')$, and $\partial_0 W \simeq f^{-1}(-\varepsilon)$, $\partial_1 W \simeq f^{-1}(\varepsilon)$ under $\phi^{\pm(\sqrt{e}-1)}$. □

Let $(\mathcal{C}, f)$ be an elementary cobordism of index λ, with characteristic and cocharacteristic embeddings φ, Φ, respectively.

Definition 18. *The* **core disk** $\mathrm{Core}_\lambda(p)$ *of the critical point p is the union of trajectories of ∇f beginning in $\varphi(\mathbb{S}^{\lambda-1}) \subset \partial_0 W$ and ending at p.*
Its **cocore disk** $\mathrm{Cocore}^{m-\lambda}(p)$ *is the union of trajectories of ∇f beginning in p and ending in $\Phi(\mathbb{S}^{m-\lambda}) \subset \partial_1 W$.*

Note that it follows from the above discussion that these are *smoothly* embedded disks, meeting transversally at p, and determining the decomposition

$$T_p W = T_p \,\mathrm{Core}_\lambda(p) \oplus T_p \,\mathrm{Cocore}^{m-\lambda}(p)$$

into negative-definite and positive-definite subpaces for $\mathrm{Hess}_p(f)$.

Corollary 5. *If $(\mathcal{C}, f)$ be an elementary cobordism of index λ,*

$$(\partial_0 W \cup \mathrm{Core}_\lambda(p)) \hookrightarrow \partial_1 W$$

is a deformation retraction. In particular

$$H_\bullet(W, \partial_0 W; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = \lambda; \\ 0 & \text{otherwise.} \end{cases}.$$

and so, in particular, the index of an elementary cobordism $\mathcal{C}$ is independent of the choice of elementary Morse function.

Corollary 6. *Every compact manifold M has the homotopy type of a finite CW-complex.*

Proof. Follows from the previous corollary, and the following fact : given a homotopy equivalence $h : X \to Y$ between topological spaces, and any $f : \partial\mathbb{D}^k \to X$, there is a homotopy equivalence

$$H : X \cup_f \mathbb{D}^k \to Y \cup_{hf} \mathbb{D}^k$$

extending h. □

In particular, $H_*(M)$ is *finitely generated.*

4.2.1. *Exercises.*

(1) A **gradient-like** vector field for $f \in \mathrm{Morse}(M)$ is a $w \in \mathfrak{X}(M)$ such that :
- $wf > 0$ on $M \setminus \mathrm{Crit}(f)$;
- For every $p \in \mathrm{Crit}(f)$, there is a Morse chart $e : B_{2\varepsilon} \hookrightarrow M$ centred at p, pulling w back to

$$e^* w = -2 \sum_1^\lambda x_i \frac{\partial}{\partial x_i} + 2 \sum_{\lambda+1}^n y_i \frac{\partial}{\partial y_i}.$$

(a) Convince yourself that, except for Lemma 4, all arguments involving the gradient ∇f with respect to some Riemannian metric remain true if ∇f is replaced by a gradient-like vector field w.

(b) Let w be a gradient-like vector field for f Morse on the *compact* manifold M, and let $\varphi : M \times \mathbb{R} \to M$ denote its flow. For any $x \in M$, let $\omega(x)$ be the collection of those points of M which are limit points sequences of the form $(\phi^{t_n}(x))_{n\geqslant 0}$, where $t_n \to +\infty$. Show that $\omega(x)$ is contained in a level set of f. Similarly, the limit points $\alpha(x)$ to sequences of the form $(\phi^{t_n}(x))_{n\geqslant 0}$, $t_n \to -\infty$, lie in a single level set of f.

(c) Show that $\alpha(x)$ and $\omega(x)$ are invariant under the flow of w.

(d) Show that $\alpha(x) \subset \mathrm{Crit}(f) \supset \omega(x)$.

(e) Show that $\alpha(x) = \{p\}$ and $\omega(x) = \{q\}$. Conclude that, for every $x \in M$, $\lim\limits_{t\to\pm\infty} \phi^t(x)$ exists and is a critical point.

(2) Prove Corollary 5.

5. Lecture Five. Morse Inequalities.

Let M be a compact manifold, and $\Bbbk$ a field (usually $\mathbb{Z}_2, \mathbb{Q}$ or $\mathbb{R}$).

Definition 19. *The* **Poincaré polynomial** *of* $f \in \mathrm{Morse}(M)$ *is*

$$P_f(t) \in \mathbb{Z}[t], \quad P_f(t) = \sum_{p \in \mathrm{Crit}(f)} t^{\lambda(f,p)} = \sum_{\lambda \geqslant 0} \mu(f, \lambda) t^\lambda.$$

The **Hilbert polynomial** *of* M *with respect to* $\Bbbk$ *is*

$$P_M(t) \in \mathbb{Z}[t], \quad P_M(t) = \sum_{\lambda \geqslant 0} \dim_\Bbbk H_\lambda(M; \Bbbk) t^\lambda.$$

Observe that both are polynomials with non-negative integer coefficients, and that

$$P_f(1) = |\operatorname{Crit}(f)|, \quad P_M(-1) = \chi(M) = \text{ Euler characteristic of } M.$$

Consider the following two orderings on polynomials in integer coefficients :

$$f \prec g \quad \Longleftrightarrow \quad g - f \in \mathbb{Z}_+[t],$$

and

$$f \preccurlyeq g \quad \Longleftrightarrow \quad g - f = (1+t)h, \quad h \in \mathbb{Z}_+[t].$$

If we express $f = \sum_\lambda f_\lambda t^\lambda, \quad g = \sum_\lambda g_\lambda t^\lambda$, then

$$f \preccurlyeq g \iff \sum_\lambda \left(\sum_{k=0}^{\lambda} (-1)^{k+\lambda} (g_\lambda - f_\lambda) \right) t^\lambda \in \mathbb{Z}_+[t].$$

From this last formula, it is clear that $f \preccurlyeq g$ implies $f \prec g$, but not conversely. (E.g., take $g = 1+t$, $f = t$.)

We wish to prove :

Theorem 13 (Morse inequalities)**.** *For every field of coefficients* $\Bbbk$*, and every Morse function* $f \in \mathrm{Morse}(M)$*, we have* $p_M \preccurlyeq p_f$*.*

Proof. Since p_f is stable under small perturbations of f, we can assume wlog that f is non-resonant. Hence we may factorize M into elementary cobordisms $M \simeq W_0 \cup \cdots W_k$. Suppose we have shown that $p_{W_0 \cup \cdots W_i} \preccurlyeq p_{f|_{W_0 \cup \cdots W_i}}$ for all $i \leqslant j$. Write

$$p_{f|_{W_0 \cup \cdots W_{i+1}}} - p_{W_0 \cup \cdots W_{i+1}} = \Big(p_{f|_{W_0 \cup \cdots W_{i+1}}} - p_{f|_{W_0 \cup \cdots W_i}} \Big) + \\ + \Big(p_{f|_{W_0 \cup \cdots W_i}} - p_{W_0 \cup \cdots W_i} \Big) + \big(p_{W_0 \cup \cdots W_i} - p_{W_0 \cup \cdots W_{i+1}} \big)$$

By definition,

$$p_{f|_{W_0 \cup \cdots W_{i+1}}} - p_{f|_{W_0 \cup \cdots W_i}} = t^\lambda,$$

where λ is the index of the unique critical point W_{i+1}.

The second term in the sum is by hypothesis of the form $(1+t)h$, $h \in \mathbb{Z}_+[t]$.

To analyze the third term, note that

$$H_\bullet(W_0 \cup \cdots W_{i+1}, W_0 \cup \cdots W_i; \Bbbk) \simeq H_\bullet(W_0 \cup \cdots W_{i+1}, \partial_0 W_{i+1}; \Bbbk)$$

and write the long exact sequence of the pair $(W_{i+1}, \partial_0 W_{i+1})$:

$$\cdots \to H_{k+1}(W_{i+1}, \partial_0 W_{i+1}) \to H_k(\partial_0 W_{i+1}) \to H_k(W_{i+1}) \to H_k(W_{i+1} \partial_0 W_{i+1}) \to \cdots$$

This shows that

$$H_k(W_{i+1}; \Bbbk) \simeq H_k(\partial_0 W_{i+1}; \Bbbk), \quad k \neq \lambda, \lambda - 1.$$

The non-trivial portion of the long exact sequence is

$$0 \to H_\lambda(\partial_0 W_{i+1}) \to\to H_\lambda(W_{i+1}) \to$$
$$\to H_\lambda(W_{i+1}, \partial_0 W_{i+1}) \xrightarrow{\delta} H_{\lambda-1}(\partial_0 W_{i+1}) \to H_{\lambda-1}(W_{i+1}) \to 0$$

Since $\simeq H_\bullet(\partial_0 W_{i+1} \cup \mathrm{Core}^\lambda, \partial_0 W_{i+1}; \Bbbk)$, and so is generated by the class

$$[\mathrm{Core}^\lambda] \in H_\lambda(W_{i+1}, \partial_0 W_{i+1})$$

There are two cases to consider :

(1) $\delta[\mathrm{Core}^\lambda] = 0$ in $H_\lambda(\partial_0 W_{i+1})$:
   Then $H_{\lambda-1}(W_{i+1}; \Bbbk) \simeq H_{\lambda-1}(\partial_0 W_{i+1}; \Bbbk)$, and
   $$0 \to H_\lambda(\partial_0 W_{i+1}) \to H_\lambda(W_{i+1}) \to \Bbbk \to 0$$
   is exact.

(2) $\delta[\mathrm{Core}^\lambda] \neq 0$ in $H_\lambda(\partial_0 W_{i+1})$:
   Then $H_\lambda(W_{i+1}; \Bbbk) \simeq H_\lambda(\partial_0 W_{i+1}; \Bbbk)$, and
   $$0 \to \Bbbk \to H_{\lambda-1}(\partial_0 W_{i+1}) \to H_{\lambda-1}(W_{i+1}) \to 0$$
   is exact.

In the first case, we have $p_{W_0 \cup \cdots W_{i+1}} - p_{W_0 \cup \cdots W_i} = t^\lambda$, while in the second, $p_{W_0 \cup \cdots W_{i+1}} - p_{W_0 \cup \cdots W_i} = -t^{\lambda-1}$. So the resulting difference is

$$p_{f|_{W_0 \cup \cdots W_{i+1}}} - p_{W_0 \cup \cdots W_{i+1}} = \begin{cases} (1+t)h & \text{if } \delta[\mathrm{Core}^\lambda] = 0 \\ (1+t)h + (1+t)t^{\lambda-1} = (1+t)(h + t^{\lambda-1}) & \text{if } \delta[\mathrm{Core}^\lambda] \neq 0. \end{cases}$$

$\square$

Definition 20. *A Morse function is called* **$\Bbbk$-completable** *if the core sphere $[\partial\, \mathrm{Core}^{\lambda(f,p)}]$ of each critical point $p \in \mathrm{Crit}(f)$ bounds. It is called* **$\Bbbk$-perfect** *if $p_f = p_M$.*

Proposition 1.

(1) A **$\Bbbk$-completable** *Morse function on a compact manifold is $\Bbbk$-perfect.*
(2) A Morse function on a compact manifold whose critical points assume no two consecutive indices is $\Bbbk$-perfect.

Example 7. *The height function f_n on $\mathbb{S}^n$ has $p_{f_n} = 1 + t^n$, so it is perfect.*
$p_{f_n \times f_m} = p_{f_n} p_{f_m} = 1 + t^n + t^m + t^{n+m}$, so if $|n - m| > 1$, $f_n \times f_m$ is also perfect.
On $\mathbb{C}P^n$, there is a perfect Morse function . For example,

$$f : \mathbb{C}^{n+1} \supset \mathbb{S}^{2n+1} \to \mathbb{R}, \quad f(z_0, ..., z_n) = \sum_1^n k|z_k|^2$$

is $\mathbb{S}^1$-invariant, and so descends to a function on $\mathbb{C}P^n$. On the affine chart $U_j = \{[z_0, ..., z_n] : z_j \neq 0$, we express f in terms of $(w_0, ..., \widehat{w_j}, ..., w_n) \in \mathbb{C}^n$, $w_k = x_k + iy_k = \frac{z_i}{z_j}$ for $k \neq j$, we have

$$f(z_0, ..., z_n) = \sum_1^n k|z_k|^2 \quad \Longrightarrow \quad f|_{U_j} = j + \sum_{k \neq j} (k - j)(x_k^2 + y_k^2).$$

Hence $[0, ..., 0, 1, 0, ..., 0]$ (with 1 in the jth position) is the unique critical point p_j of f on U_j; by inspection one sees that $\mathrm{Hess}(f)$ has eigenvalues $(0 - j, 0 - j, 1 - j, 1 - j, ..., n - j, n - j)$ (with $(j - j, j - j)$ removed), and so $\lambda(f, p_j) = 2j$. Thus $p_f = \sum_0^n t^{2j}$, and so f is perfect, and gives

$$H_\bullet(\mathbb{C}P^n) = \mathbb{Z}[t^2]/(t^{2n+2}].$$

5.0.2. *Exercises.*

(1) Show that the height function $f_n : \mathbb{S}^n \to \mathbb{R}$ is $\mathbb{Q}$-perfect. If $|n-m| > 1$, then $(x, y) \mapsto f_n(x)f_m(y)$ is $\mathbb{Q}$-perfect, and compute $H_\bullet(\mathbb{S}^n \times \mathbb{S}^m; \mathbb{Q})$.

(2) Let $f : \mathbb{S}^{2n+1} \to \mathbb{R}$ be given by

$$F : \mathbb{C}^{n+1} \supset \mathbb{S}^{2n+1} \to \mathbb{R}, \quad F(z_0, ..., z_n) = \sum_1^n k|z_k|^2.$$

   (a) Show that F is invariant under the action of $\mathbb{S}^1$ on $\mathbb{C}^{n+1}$, and thus descends to $f :\in C^\infty(\mathbb{C}P^n)$.

   (b) Show that $\mathrm{Crit}(f) = \{p_1, ..., p_n\}$, $p_i = [0, ..., 0, 1, 0., , , .0]$ (ith position), and show that f is Morse.

   (c) Show that the index of f at p_j is $2j$. Conclude that f is $\mathbb{Q}$-perfect, and compute $H_\bullet(\mathbb{C}P^n; \mathbb{Q})$.

(3) Show that if M is compact and of odd dimension, its Euler characteristic $\chi(M)$ vanishes.

(4) Knowing that $H_k(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2$ for all $0 \leqslant k \leqslant n$, conclude that the least number of critical points of a Morse function on $\mathbb{R}P^n$ is $n + 1$.

(5) If a Morse function on a compact M^m has exactly three critical points, then their indices are $0, m/2, m$. In particular, M has even dimension.

6. Lecture Six. The Morse-Smale condition and genericity.

Let (W, M_{-1}, M_1) be a cobordism, $f : W \to \mathbb{D}^1$ a Morse function. We have seen earlier how, upon perturbing f ever so slightly, we can arrange that no two critical points lie at the same regular set. We then saw in Lemma 12 that this allowed us to factor any cobordism as a composition of elementary cobordisms, $W \simeq W_0 \cup \cdots \cup W_k$.

In this lecture we introduce the problem of *rearrangements* of cobordisms. The model situation is the following : suppose W is a composition of two cobordisms $W = W_0 \cup W_1$, where W_i only contains critical points of index λ_i. Can we express W as a composition of cobordisms $W_0' \cup W_1'$, where W_0' only contains points of index λ_1 and W_1' of index λ_0 ? (It is useful to keep in perspective that CW complexes are defined by successively attaching cells in *increasing* order of dimension).

We saw in Lecture 4 that if (W, M_{-1}, M_1) is an elementary cobordism, it is diffeomorphic to the trace of a surgery on M_{-1} corresponding to some embedding $\varphi : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \hookrightarrow M_{-1}$, and we concluded that $W \simeq M_{-1} \times [-1, +1] \cup L_\lambda$, the attachment of a λ-handle L_λ to the trivial cobordism $M_{-1} \times [-1, +1]$. Of course, the biggest difficulty in the rearrangement business is that the next handle we attach to $M_1 \times [-1, 1]$ in general intersects the previous handle. However, when 'the handles do not meet', we can rearrange the cobordisms.

Theorem 14. *Any cobordism (W, M_{-1}, M_1) can be written as a composition $W = W_0 \cup \cdots \cup W_{m+1}$, where W_i only contains critical points of index i.*

We will prove something slightly stronger.

Definition 21. *A Morse function f on a cobordism (W, M_0, M_1) is called* **nice** *if $f : W \to [-\frac{1}{2}, m + \frac{1}{2}]$, and all critical points of index i of f lie at the level set $f = i$.*

Theorem 15. *Every cobordism has a nice Morse function.*

The path to prove this theorem is of independent interest.

First, we need to be able to be very definite about the critical values of our critical points. The simplest case is when f has two (sets of) critical points p, p' lying on two interior level sets $f = a$, $f = a'$, possibly the same. We would like to modify f into some new Morse function g so that p, p' lie now at $g = b$ and $g = b'$, for a priori given $-1 < b, b' < 1$. It turns out that this is possible to accomplish if 'no critical trajectories of p intersect'.

Definition 22. *A* **gradient pair** *is a pair (f, v), where f is a Morse function, and v is a gradient-like vector field for f.*

If (f, v) is a gradient pair, and $x \in \mathrm{Crit}(f)$, in a neighborhood of x v looks like

$$v = -2\frac{\partial}{\partial x_1} - \cdots - 2\frac{\partial}{\partial x_\lambda} + 2\frac{\partial}{\partial x_{\lambda+1}} + \cdots + 2\frac{\partial}{\partial x_m} \in \mathfrak{X}(\mathbb{R}^{m+1});$$

observe that if ϕ_t denotes the flow of the opposite of the vector field above, those points $y \in \mathbb{R}^{m+1}$ for which $\phi_t(x) \to x$ as $t \to -\infty$ are precisely $\mathbb{R}^\lambda \times 0$, while those for which $\phi_t(y) \to x$ as $t \to +\infty$ are exactly $0 \times \mathbb{R}^{m+1-\lambda}$.

Definition 23. *The* **stable manifold** $W^s(x; v)$ *of $x \in \mathrm{Crit}(f)$ is the subspace*

$$W^s(x; v) := \{y : \phi_t(y) \to x \text{ as } t \to +\infty\}$$

The **unstable manifold** $W^u(x; v)$ *of x is*

$$W^u(x; v) := \{y : \phi_t(y) \to x \text{ as } t \to -\infty\}$$

The terminology is justified for the following reason :

Lemma 16. *If M is compact, (f, v) a gradient pair, and $x \in \operatorname{Crit}(f)$, $W^u(x; v)$ and $W^s(x; v)$ are the image of smooth, injective immersions*

$$E^u : T_x^u M \to M, \quad E^s : T_x^s M \to M.$$

Proof Sketch. We have a model for the singularities, and around x $W^u(x; v)$ meets the boundary of a small disc $D \subset M$ centred at x in a $(\lambda - 1)$-sphere, which we identify with the unit sphere in $T_x^u M$. We can now check directly that the formula $w \mapsto \phi_{\frac{1}{2}\log|w|}(\frac{w}{|w|})$ is smooth at x, and injectively immerses $T_x^u M$ into M. □

Together with Exercise 1 of Lecture 4, we conclude that the open cells of (un)stable manifolds cover the whole manifold :

$$M = \coprod_{\operatorname{Crit}(f)} W^u(x; v) = \coprod_{\operatorname{Crit}(f)} W^s(x; v)$$

Let v be a gradient-like vector field for f, and let $K(x) \subset W$ denote the $W^u(x; v) \cup W^s(x; v)$. Note that v vanishes nowhere on $M \setminus \cup_{\operatorname{Crit}(f)} K(x)$, and the latter is saturated by trajectories of v.

Lemma 17. *If* $\operatorname{Crit}(f) = \operatorname{Crit}_- \coprod \operatorname{Crit}_+$, *where* $f \operatorname{Crit}_- = \{a_-\}$, $f \operatorname{Crit}_+ = \{a_+\}$, *and*

$$x \in \operatorname{Crit}_-, y \in \operatorname{Crit}_+ \quad \Longrightarrow \quad K(x) \cap K(y) = \varnothing,$$

then given $-1 \leqslant b, b' \leqslant 1$, there is a Morse function $\widetilde{f} : W \to \mathbb{D}^1$ for which ξ is still gradient-like, and such that :

(1) *agreeing with f around ∂W,*
(2) *having the same critical points as f,*
(3) $d\widetilde{f} = df$ *around* $\operatorname{Crit}(f)$,
(4) $\widetilde{f} \operatorname{Crit}_- = b$, $\widetilde{f} \operatorname{Crit}_+ = b'$.

Proof. Let $K_\pm := \cup_{\operatorname{Crit}_\pm} K(x)$; by hypothesis $K_- \cap K_+ = \varnothing$. Let $\varrho' : M_{-1} \to \mathbb{D}^1$ be a smooth function with

$$\varrho'|_{\operatorname{Op} K_- \cap M_{-1}} = -1, \quad \varrho'|_{\operatorname{Op} K_+ \cap M_{-1}} = 1.$$

Then there exists a unique extension $\varrho : W \to \mathbb{D}^1$ of ϱ' which is constant on trajectories of ξ, that is, $d\varrho(\xi) = 0$.

Consider now a smooth map $H : \mathbb{D}^1 \times \mathbb{D}^1 \to \mathbb{D}^1$, satisfying

(1) $H(x, y) = x$, for all $x \in \operatorname{Op} \partial\mathbb{D}^1$ and all $y \in \mathbb{D}^1$;
(2) $\frac{\partial H}{\partial x}(x, y) = 1$ for $x \in \operatorname{Op} \operatorname{Crit}(f)$ and $y \in \mathbb{D}^1$, and $\frac{\partial H}{\partial x}(x, y) > 0$ for all $x, y \in \mathbb{D}^1$;
(3) $H(a_-, -1) = b$, $H(a_+, 1) = b'$,

and set $\widetilde{f} := H \circ (f \times \varrho)$.

Observe that $d\widetilde{f} = d^1 H \circ df + d^2 H \circ d\varrho$. Outside $\operatorname{Crit}(f)$, we have $df(\xi) > 0$ and by condition (2) and $d\varrho(\xi) = 0$, $d\widetilde{f}(\xi) > 0$ outside $\operatorname{Crit}(f)$; in particular, $\operatorname{Crit}(\widetilde{f}) \subset \operatorname{Crit}(f)$. Now, by construction,$d\varrho$ vanishes around $\operatorname{Crit}(f)$, so $d\widetilde{f} = df$ around each critical point, and in particular $\operatorname{Crit}(\widetilde{f}) = \operatorname{Crit}(f)$. Note also that condition (1) ensures that $\widetilde{f} = f$ around ∂W, and that

$$x \in \operatorname{Crit}_- \Rightarrow \widetilde{f}(x) = H(a_-, -1) = b, \quad x \in \operatorname{Crit}_+ \Rightarrow \widetilde{f}(x) = H(a_+, 1) = b'$$

□

Definition 24. *A gradient pair (f, v) is called* **Morse-Smale** *if, for every two critical points $p, q \in \operatorname{Crit}(f)$, we have*

$$W^u(q) \pitchfork W^s(p).$$

There are two crucial observations to be made :

- The intersection $W^u(q) \cap W^s(p)$ consists of trajectories of ϕ_t going from q to p. Since $t \mapsto f(\phi_t(x))$ is decreasing for every x, and strictly so if $x \notin \mathrm{Crit}(f)$, we see that
$$W^u(q) \cap W^s(p) = \varnothing$$
if $f(q) \leqslant f(p)$. So these are all transverse, and we may restrict our attention to $W^u(q) \cap W^s(p)$, for $f(q) > f(p)$.
- The condition $W^u(q) \pitchfork W^s(p)$ holds iff, for some regular value $c \in (f(p), f(q))$, we have
$$W^u(q) \cap f^{-1}(c) \pitchfork W^s(p) \cap f^{-1}(c).$$

Lemma 18. *Let $f : W \to \mathbb{D}^1$ be a Morse function, (f, v) a gradient pair, and suppose $[a, b] \subset \mathring{\mathbb{D}}^1$ contains no critical values. Let h_t be an isotopy of $f^{-1}(b)$.*

There exists then a gradient-like vector field $\widetilde{v}$ for f, such that

(1) $\widetilde{v}$ agrees with v outside $f^{-1}(a, b)$;
(2) the diffeomorphisms $\varphi, \widetilde{\varphi} : f^{-1}(a) \to f^{-1}(b)$ induced by the flows of $v, \widetilde{v}$ are related by $\widetilde{\varphi} = h_1\varphi$.

Proof. Reparametrizing h_t, we can assume it is stationary around $0, 1$.

The flow ϕ_t of $\frac{v}{vf} =: \widehat{v}$ defines a diffeomorphism $\phi : [a, b] \times f^{-1}(b) \xrightarrow{\sim} f^{-1}[a, b]$, where $f(\phi_t(x)) = t$. Define
$$H : [a, b] \times f^{-1}(b) \xrightarrow{\sim} [a, b] \times f^{-1}(b), \quad H(t, x) := (t, h_t(x)),$$
and set $\widehat{w} := (\phi H \phi^{-1})_* \widehat{v}$. Note that it is smooth on $f^{-1}[a.b]$, agrees with $\widehat{v}$ near $f^{-1}(a)$, and $\widehat{w} f = 1$. Hence
$$w := \begin{cases} v & \text{outside } f^{-1}[a, b]; \\ (vf)\widehat{w} & \text{on } f^{-1}[a, b] \end{cases}$$
defines a gradient-like vector field for f with the required properties. $\square$

Theorem 16. *Let M be compact and $f \in \mathrm{Morse}_{\neq}(M)$. Then there exists a Morse-Smale pair (f, v).*

Proof. Choose a gradient-like vector field v and order the critical values of f,
$$f\,\mathrm{Crit}(f) = c_1 < c_2 \cdots < c_N,$$
and let $p_i \in \mathrm{Crit}(f) \cap f^{-1}(c_i)$. As we observed before, $W^u(p_j) \cap W^s(p_i) \neq \varnothing$ only if $f(p_j) = c_j > c_i = f(p_i)$.

Suppose then that, for a fixed $1 \leqslant k \leqslant N$, we knew that
$$i, j < k \quad \Longrightarrow \quad W^u(p_i) \pitchfork W^s(p_j).$$
Choose $[a, b] \subset (c_{k-1}, c_k)$, and let $V_a := f^{-1}(a)$, $V_b := f^{-1}(b)$. Note that, if $W^u(p_k) \cap V$ were transverse to each $W^s(p_i) \cap V_b$, $i < k$, we would be done.

Suppose this is not the case. We seek then to modify the gradient-like vector field v so that the above intersections be transverse. For this purpose, choose an isotopy $h_t : V_b \to V_b$, with the property that
$$h_1\left(\bigcup_1^{k-1} W^s(p_i; v)\right) \cap V_b \pitchfork W^u(p_k; v) \cap V_b.$$
Use Lemma 18 to produce another gradient-like vector field w for f, agreeing with v outside $a < f < b$, and whose corresponding diffeo $\phi^w_{a-b} : V_a \xrightarrow{\sim} V_b$ reads

$h_1\phi^v_{a-b}|_{V_a}$. Note that the first condition says that we do not disturb the lower ascending spheres, and the upper descending sphere :

$$W^s(p_i;w)\cap V_a = W^s(p_i;v)\cap V_a, \quad i<k,$$
$$W^u(p_k;w)\cap V_b = W^s(p_k;v)\cap V_b.$$

Moreover, by the second property, we have for $i<k$

$$W^s(p_i;w)\cap V_b = \phi^w_{a-b}\left(W^s(p_i;w)\cap V_a\right) = h_1\phi^w_{a-b}\left(W^s(p_i;v)\cap V_a\right) = h_1\left(W^s(p_i;v)\cap V_b\right),$$

so

$$W^u(p_k;w) \pitchfork W^s(p_i;w), \quad i<k+1.$$

Proceeding inductively, we arrive at the desired conclusion. $\square$

Corollary 7. *If (f,v) is Morse-Smale, and $W^u(q)\cap W^s(p)\neq\varnothing$, then $\lambda(f,p)<\lambda(f,q)$.*

Theorem 17. *If $f: W\to\mathbb{D}^1$ is a Morse function, there is $\widetilde{f}\in \mathrm{Morse}(W)$, $\widetilde{f}: W\to[\frac{1}{2}, m+\frac{3}{2}]$, such that*

(1) $\mathrm{Crit}(\widetilde{f}) = \mathrm{Crit}(f)$*;*
(2) $\lambda(\widetilde{f},p)=\lambda(f,p)$ *for all* $p\in\mathrm{Crit}(f)$*;*
(3) $\widetilde{f}(p)=\lambda(\widetilde{f},p)$.

Proof. Perturb f to Morse non-resonant, and choose a Morse-Smale pair (v,f). If p_i,p_{i+1} are to successive critical points of f – that is, if $(f(p_i),f(p_{i+1}))$ is non-empty and contains no critical values, and $\lambda(f,p)>\lambda(f,q)$, then critical trajectories do not meet, and we can apply Lemma 17 to modify f to some Morse function $\widetilde{f}$, agreeing with f on the complement of the interior of $W_i\cup W_{i+1}$, in such a way that $\widetilde{f}(p_{i+1})<\widetilde{f}(p_i)$. In that manner, we can assume without loss of generality that the critical points of f occurr with *increasing* indices; that is, we can rearrange W as

$$W = W_0\cup W_1\cup\cdots\cup W_{m+1},$$

where all critical points of index λ are contained in W_λ. Apply Lemma 17 repeatedly to each W_λ until all critical points lie in the same level set, and compose with the appropriate diffeomorphism $f(M)\xrightarrow{\sim}[-\frac{1}{2},m+\frac{3}{2}]$. $\square$

Definition 25. *A Morse function f satisfying $f(p)=\lambda(f,p)$ for all $p\in\mathrm{Crit}(f)$ is called* **nice**, *or* **self-indexing**.

Corollary 8 (First Rearrangement Theorem)**.** *Any handle attachment can be performed in increasing dimension.*

6.1. Exercises.

(1) Show that the standard height function of the upstanding torus, with the gradient vector field induced by the standard metric in $\mathbb{R}^3$, is *not* Morse-Smale.
(2) Give an example of a gradient pair (f,v) for which the closure of some unstable manifold is *not* homeomorphic to a closed disk.

7. Lecture Seven : The Cancellation Theorem

Let $f : W \to \mathbb{D}^1$ be a Morse function, v a gradient-like vector field, having as unique critical points p, q, of indices $\lambda, \lambda+1$, with $f(p) < c < f(q)$, and let $V := f^{-1}(c)$.

Theorem 18 (Cancellation Theorem)**.** *If $W^u(q; v) \cap V$ intersects $W^s(p; v) \cap V$ transversally at a single point, it is possible to alter v in a neighborhood of the unique critical trajectory $q \to p$ to a vector field, all of whose trajectories go from $\partial_0 W$ to $\partial_1 W$, which is gradient-like for a Morse function $\widetilde{f}$ without critical points, which agrees with f around ∂W.*

Suppose $\gamma \subset W$ is the critical trajectory $p \to q$ of the gradient-like vector field v. Let us say that a gradient pair (v, f) is **good** if there exist a neighboorood $U(\gamma) \subset W$ of γ and an open embedding $\varphi : U(\gamma) \to \mathbb{R}^{m+1}$, such that

(1) $\varphi(p) = (0, ..., 0)$, $\varphi(q) = (1, ..., 0)$;
(2) $\varphi_* v = u$, where

$$u = a(x_1)\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} - \cdots \frac{\partial}{\partial x_{\lambda+1}} + \frac{\partial}{\partial x_{\lambda+2}} + \cdots + \frac{\partial}{\partial x_{m+1}}$$

and a is a smooth function with

$$a(t) > 0 \text{ for } 0 < t < 1, \quad a(0) = 0 = a(1)$$

$$a'(t) = t \text{ on } \operatorname{Op} 0, \quad a'(t) = -t \text{ on } \operatorname{Op} 1, \quad 2\int_0^1 a(t)dt = f(q) - f(p).$$

Lemma 19. *Under the hypotheses of Theorem 18, we can modify v around f so as that (f, v) be good.*

Proof. Consider the function $F : \mathbb{R}^{m+1} \to \mathbb{R}$,

$$F(x_1, ..., x_{m+1}) = f(p) + 2\int_0^{x_1} a(t)dt - x_1^2 - \cdots - x_{\lambda+1}^2 + x_{\lambda+2}^2 + \cdots + x_{m+1}^2,$$

which is Morse and has u as a gradient-like vector-field. Note that $F(0) = f(p)$ and $F(e) = f(q)$, where $e = (1, 0, ..., 0)$.

Let $\widetilde{\gamma} := \{(t, 0, ..., 0) : 0 \leqslant t \leqslant 1\}$, choose regular values $f(p) < b_1 < b_2 < f(q)$, and on small, disjoiint neighborhoods U_0 and U_2 of $0, e$, choose embeddings $\varphi_i : U_i \hookrightarrow W$, $i = 0, 2$, where , and $\varphi_0(0) = p$, $\varphi_2(q) = e$ and $\varphi_i^* f = F$, $\varphi_i^* v = u$. Now, u induces an isotopy of a small neighborhood of $\widetilde{\gamma} \cap \{F = b_1\}$ to a small neighborhood of $\widetilde{\gamma} \cap \{F = b_2\}$, mapping $\widetilde{\gamma} \cap \{F = b_1\}$ to $\widetilde{\gamma} \cap \{F = b_2\}$; let then $\varphi_{01} : \operatorname{Op} \widetilde{\gamma} \cap \{F \leqslant b_2\} \hookrightarrow W$ be the induced embedding. Note that φ_{01} is uniquely determined by requiring that it extend φ_0 and maps trajectories of u to trajectories of v, and level sets of F to level sets of f.

If φ_{01} and φ_2 agree around $\widetilde{\gamma} \cap \{F = b_2\}$, they glue into an embedding $\varphi : U(\widetilde{\gamma}) \hookrightarrow W$ preserving level sets and trajectories, and so

$$\varphi_* u = e^h v, \quad h \in C^\infty(W).$$

Then $(f, \widetilde{v})$ is good, where $\widetilde{v} := e^h v$.

We conclude by observing that we can always make φ_{01} and φ_2 agree around $\widetilde{\gamma} \cap \{F = b_2\}$, since they are isotopic there, and thus, at the expense of changing v around $\widetilde{\gamma} \cap \{b_1 < F < b_2\}$, Lemma 18 applies. ☐

According to the Lemma, we can assume wlog that (f, v) is good. Let $\varphi : U(\gamma) \hookrightarrow \mathbb{R}^{m+1}$ the embedding in the definition of 'good'.

Lemma 20. *There exist opens $U' \subset U \subset U(\gamma)$ around γ, $\operatorname{Cl} U' \subset U$, such that no trajectory of $\widetilde{v}$ which passes through U' and goes outside U comes back inside U'.*

Proof. Choose any U, and suppose no U' verified the claim. Choose then a sequence U'_n of neighborhoods, with $\gamma = \bigcap U'_n$; then for each n, there must be a trajectory c_n connecting point $x_n, z_n \in U'_n$ through a point $y_n \in W \diagdown U$, with

$$\operatorname{dist}(x_n, \gamma) \to 0, \quad \operatorname{dist}(z_n, \gamma) \to 0.$$

Since $W \diagdown U$ is compact, we may assume that $y_n \to y$. Now, y either comes from $\partial_0 W$ or goes to $\partial_1 W$, since the only other option would be for it to be a trajectory from p to q, and this is not the case since $y \notin U$. Suppose y comes from $\partial_0 W$. Then (by continuity of the flow), every point y' sufficiently close to y also comes from $\partial_0 W$. Now, for any such point, the trajectory from $\partial_0 W$ to y' is compact, and so the distance $\operatorname{dist}(\gamma, \gamma(y'))$ between γ and the trajectory that contains y' is bounded away from zero in a small neighborhood of y. But $y_n \to y$ and $\gamma(y_n)$ contains x_n, contradicting $\operatorname{dist}(\gamma, x_n) \to 0$. □

Proof of Theorem. We will break it down into a few steps, following Milnor as we have been for a while now.

STEP ONE. With the same notation as in the previous Lemma, we claim that there is a nowhere-vanishing vector field $\widetilde{v}$ on W, which agrees with v outside some compact $\gamma \subset K \subset U'$, with the property that every trajectory of $\widetilde{v}$ was outside of U at some negative time, and will go outside of U at some positive time.

To achieve this, let $\widetilde{a} : \mathbb{R} \times \mathbb{R}$ be a smooth function, with $\widetilde{a}(t,s) = a(t)$ for s away from zero, and $\widetilde{a}(t,s) < \alpha < 0$ for $|s| \leqslant \varepsilon$. Consider

$$\widetilde{u} := \widetilde{a}\left(x_1, r(x_2, ..., x_{m+1})\right) \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} - \cdots \frac{\partial}{\partial x_{\lambda+1}} + \frac{\partial}{\partial x_{\lambda+2}} + \cdots + \frac{\partial}{\partial x_{m+1}},$$

where $r(x_2, ..., x_{m+1}) = x_2^2 + \cdots + x_{m+1}^2$.

Note that, by construction, $\varphi^* \widetilde{u}$ defines a non-vanishing vector field $\widetilde{v}$ on W; taking $\varepsilon > 0$ small enough, $\widetilde{v}$ will agree with v outside some compact $K \subset U$.

Let $c(t)$ denote the trajectory of $\widetilde{u}$ with initial condition $\varphi(x) = (\overline{x}_1, ..., \overline{x}_m)$. If some $\overline{x}_i$, $\lambda + 2 \leqslant i \leqslant m+1$ is non-zero, then $|c(t)|$ increases exponentially, and so the trajectory of $\widetilde{v}$ through x must leave U. Suppose on the other hand that $\overline{x}_i = 0$ for all $\lambda + 2 \leqslant i \leqslant m+1$. Let us agrue by contradiction, and assume that $c(t)$ does not leave U. Then $\varrho(c(t)) = c(0)e^{-2t}$, so $\varrho(c(t)) < \varepsilon$ for large enough t. But then $\frac{dc(t)}{dt} < \alpha$, so $c(t)$ cannot stay in U.

A symmetric argument shows that every trajectory comes from outside of U.

STEP TWO. We claim that $\widetilde{v}$ induces a diffeomorphism $\partial_0 W \times [0,1] \to W$. Note first that every trajectory of $\widetilde{v}$ goes from $\partial_0 W$ to $\partial_1 W$. Indeed, there are two cases : either the trajectory meets U' or it does not. If it doesn't, then the claim follows essentially by hypothesis. So suppose a trajectory passes through U'. Then it must leave U at some point, by Step One. Such trajectories cannot return to U', according to the previous Lemma, and so they must go from $\partial_0 W$ to $\partial_1 W$.

Let now $\tau_0, \tau_1 : W \to \mathbb{R}$ be the functions assigning to wach $x \in W$ the times $\tau_0(x)$, $\tau_1(x)$ taken by the flow of $\widetilde{v}$ to reach $\partial_0 W$, $\partial_1 W$. That is,

$$x \in \phi^{\widetilde{v}}_{\tau_0(x)}(\partial_0 W) \cap \phi^{\widetilde{v}}_{-\tau_1(x)}(\partial_1 W).$$

Then consider $p : W \to \partial_0 W$ given by $p(x) = \phi^{\widetilde{v}}_{-\tau_0(x)}(x)$, and observe that now the vector field

$$w(x) := \tau_1(p(x))\widetilde{v} \in \mathfrak{X}(W)$$

has the same trajectories as $\widetilde{v}$, which w sweeps in unit time, and so

$$\Phi := \phi^w : \partial_0 W \times [0,1] \xrightarrow{\sim} W, \quad (x,t) \mapsto \phi^w_t(x),$$

is a diffeomorphism, with inverse $x \mapsto (p(x), \tau_0(x))$.

STEP THREE.

We conclude by producing a Morse function $\widetilde{f} : W \to \mathbb{R}$, having no critical points, and w as a gradient-like vector field, and agreeing with f around ∂W. But according to Step Two, all we need to do is find a function

$$g : \partial_0 W \times [0,1] \longrightarrow \mathbb{R},$$

satisfying

$$\frac{\partial g}{\partial t} > 0, \quad g = \Phi^* f \text{ around } \partial_0 W \times \partial[0,1].$$

Since $\frac{\partial \Phi^* f}{\partial t} > 0$ along $\partial_0 W \times \partial[0,1]$, there is $\varepsilon > 0$ such that $\frac{\partial \Phi^* f}{\partial t} > 0$ on $t \in [0,\varepsilon] \coprod [1-\varepsilon, 1]$. Choose a smooth function $\varrho : [0,1] \to [0,1]$ which is zero on $[\varepsilon, 1-\varepsilon]$ and one around $\partial[0,1]$, and consider

$$k_\varepsilon(x) := \frac{1 - \int_0^1 \varrho(s) \frac{\partial \Phi^* f}{\partial s}(x,s) ds}{1 - \int_0^1 \varrho(s) dt}.$$

Then note that, for $\varepsilon > 0$ small enough, we have $k_\varepsilon > 0$ on ∂W. For such small ε, let

$$g(x,t) := \int_0^t \left[\varrho(s) \frac{\partial \Phi^* f}{\partial s}(x,s) + (1 - \varrho(s)) k_\varepsilon(x) \right] ds.$$

Then $g(x,0) = 0$, $g(x,1) = 1$ by the definition of g, and $g(x,t) = \Phi^* f$ near $\partial_0 W \times \partial[0,1]$. $\square$

Particularly useful were : Milnor, Bott, Golubitsky-Guillemin, Kosinski, Gualtieri, Nicolaescu and Lück.

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