MORSE THEORY

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1. Lecture One. Propaganda. Poincaré conjecture in dimensions $\geqslant 5$.

1.1. **Singularities of smooth maps.** Let M, M' be smooth manifolds[1], and $f : M \longrightarrow M'$ a smooth map. We define

$$\mathsf{Crit}(f) := \{x \in M : \mathsf{rank} d_x f < \min(\dim M, \dim M')\}$$

the set of **critical points** of f, and $f\mathsf{Crit}(f) \subset M'$ the set of **critical values** of f. $M \diagdown \mathsf{Crit}(f)$ and $\mathbb{R} \diagdown f\mathsf{Crit}(f)$ are then said to consist of **regular points** and **regular values**, respectively.

Note that $\mathsf{Crit}(f) \subset M$ is closed.

1.1.1. *Abundance of regular values : Sard's theorem.* Recall that a subspace $X \subset \mathbb{R}^m$ is said to have **measure zero** if, for all $\varepsilon > 0$ there is a sequence of balls $(B_n)_{n \geqslant 0}$, with

$$\sum_n \mathsf{vol}(B_n) < \varepsilon, \quad \bigcup_n B_n \supset X. \quad \mathsf{vol}(B_n) := \int_{B_n} dx$$

One immediately checks that :

- $(X_n)_{n \geqslant 0}$ have measure zero $\Longrightarrow \bigcup_n X_n$ has measure zero;
- $g : \mathbb{R}^m \to \mathbb{R}^m$ smooth, $X \subset \mathbb{R}^m$ of measure zero $\Longrightarrow g(X) \subset \mathbb{R}^m$ has measure zero.

Hence the following notion is well-defined : a subspace $X \subset M$ is said to have measure zero if there exists a smooth atlas $\mathfrak{A} = \{(U_i, \varphi_i)\}$ of M, with each

$$\varphi_i(X \cap U_i) \subset \mathbb{R}^m$$

of measure zero.

Recall now :

Theorem 1 (Sard)**.** *If $f : M \longrightarrow M'$ is smooth, $f\mathsf{Crit}(f) \subset M'$ has measure zero.*

Proof. See [9] or [29]. □

Hence 'almost all' values are regular.

When $\dim M \ll \dim M'$, there is a 'lot of space' to deform f inside M', and we can always *remove* the singularities of f – i.e., perturb it slightly to a f' with $\mathsf{Crit}(f') = \varnothing$.

Theorem 2 (Whitney)**.** *Every $f : M \to M'$ is C^∞-close to a injective immersion if $\dim M' \geqslant 2 \dim M$, and every M^m embeds in $\mathbb{R}^{2m}$.*

In the other extreme, if M is compact, without boundary, $\mathsf{Crit}(f) \neq \varnothing$.

So we cannot get rid of singularities of functions.

1.1.2. *Singularities of Functions.* We consider the assignement $M \mapsto C^\infty(M)$.

> **Meta-principle 1 : Min-Max:** "The more complicated the topology of M, the greater the number of critical points of functions on it."

Example 1. *If $f : \mathbb{T}^n \to \mathbb{R}$, then $|\mathsf{Crit}(f)| \geqslant n + 1$.*

For more, see Min-Max theory ⊛.

> **Meta-principle 2 : Morse-Smale:** "The dynamics of a nice enough $f \in C^\infty(M)$ reconstructs M smoothly."

Example 2. *Suppose M^m is compact without boundary, and $f : M \to \mathbb{R}$ has exactly two critical points. Then M^m is* homeomorphic *to S^m.*

[1]Throughout these notes, by a *manifold*, we mean a Hausdorff, second-countable topological space, equipped with a maximal smooth atlas.

[DRAWING ⊛]

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Aside : Poincaré Conjecture & Homotopy Spheres

Remark 1. *It is* not *claimed that M^m is diffeomorphic to S^m, with its standard smooth structure. In fact, in* [14]*, Milnor constructs smooth $\mathbb{S}^3$-bundles $p : M \to \mathbb{S}^4$, for which there cannot exist B^8 with*

$$\partial B = M, \quad H^4(B;\mathbb{Z}) = 0,$$

and carries a smooth $f \in C^\infty(M)$ with exactly two non-degenerate critical points. This implies that M^7 is homeomorphic to $\mathbb{S}^7$, but not diffeomorphic to it; such manifolds are called **exotic spheres***.*

Definition 1. *A* **homotopy sphere** *is a smooth, oriented manifold M^m, homotopy-equivalent to $\mathbb{S}^m$.*

Note that if M^m is a homotopy sphere, then $\pi_1(M) = \{1\}$, and $H_\bullet(M;\mathbb{Z}) \simeq H_\bullet(\mathbb{S}^m;\mathbb{Z})$. Conversely, if M^m is simply connected and $H_\bullet(M;\mathbb{Z}) \simeq H_\bullet(\mathbb{S}^m;\mathbb{Z})$, then M^m is a homotopy sphere; indeed, in that case $\pi_\bullet(M) \simeq \pi_\bullet(\mathbb{S}^m)$ by Hurewicz' theorem. Now, a generator $[\alpha] \in \pi_m(\mathbb{S}^m)$, $\alpha : \mathbb{S}^m \to M$, gives rise to a homotopy equivalence $\mathbb{S}^m \xrightarrow{\simeq} M$.

Homotopy spheres are the object of the famous

Theorem 3 (Poincaré Conjecture)**.** *A homotopy sphere M^m is homeomorphic to $\mathbb{S}^m$.*

Observe that the *smooth* version of the theorem, claiming that homotopy spheres are *diffeomorphic* to $\mathbb{S}^m$, is decidedly false in light of the existence of exotic spheres.

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Example 3. *Milnor's stand-up torus... ⊛. How does one make a drawing ?*

- *Present $(\mathbb{T}^2, f)$ and its 4 critical points;*
- *'Non-degenerate' allows* normal forms *around the points; describe them;*
- *Show how the topology of $f^{-1}(-\infty, t]$ changes as t varies. ('reconstruction').*

Note that :

(1) If $[a, b]$ contains no critical values of f, then the diffeomorphism type of $f^{-1}(-\infty, t]$ is independent of $t \in [a, b]$;

(2) When t crosses a critical value $t = c$, we have

$$f^{-1}(-\infty, c + \varepsilon] = f^{-1}(-\infty, c - \varepsilon] \cup H_\lambda,$$

where H_λ denotes a "handle" $H_\lambda \sim \mathbb{D}^\lambda \times \mathbb{D}^{m-\lambda}$.

In view of Sard's theorem, (1) suggests that we subdivide our task of understanding the topology of M.

Definition 2. *A* **cobordism** $\mathcal{C} = (W; M_0, f_0, M_1, f_1)$ *from M_0^m to M_1^m is a smooth manifold W^{m+1}, together with a decomposition of its boundary as $\partial W = \partial_0 W \coprod \partial_1 W$, together with diffeomorphisms $f_i : \partial_i W \xrightarrow{\simeq} M_i$.*

If M_i are oriented (as will usually be the case), we assume further that W is oriented, and that f_0 be orientation-*preserving*, while f_1 is orientation-*reversing*; we refer to $\mathcal{C}$ as an **oriented cobordism** between M_0 and M_1. We will also refer to $\partial_0 W$ as the **incoming** boundary of W, and to $\partial_1 W$ as the **outgoing** boundary of W.

We regard a cobordism $\mathcal{C}$ as a 'morphism' $M_0 \rightsquigarrow M_1$ of sorts. The maps will be suppressed from the notation when no confusion can arise.

Definition 3. *Two cobordisms* $\mathcal{C} = (W, M_0, f_0, M_1, f_1)$ *and* $\mathcal{C} = (W', M_0', f_0', M_1', f_1')$ *are said to be* **equivalent over** M_0 *if there exists an oriented diffeomorphism*

$$F : W \xrightarrow{\simeq} W', \quad f_0' \circ F = f_0.$$

A cobordism $\mathcal{C} = (W, M_0, f_0, M_1, f_1)$ *is called :*

- **trivial** *if it is equivalent over* M_0 *to* $(M_0 \times [0,1]; M_0, M_1)$*;*
- *an* h**-cobordism** *if* $\partial_i W \hookrightarrow W$ *are homotopy equivalences.*[2]

One of the goals of this course is to prove the following fundamental

Theorem 4 (Smale's h-cobordism theorem)**.** *If* $\pi_1 M_0 = \{1\}$ *and* $\dim M \geqslant 5$*, any* h*-cobordism over* M_0 *is trivial.*

Corollary 1 (Characterization of disks)**.** *If* M^m *is a contractible, smooth, compact manifold, and* $\pi_1(\partial M) = \{1\}$*, then* $M \simeq D^m$ *if* $m \geqslant 6$*.*

Proof. Choose an embedding $j : \mathbb{D}^m \hookrightarrow M^m$, and let

$$\widehat{M} := M \diagdown j(\overset{\circ}{\mathbb{D}}{}^m).$$

Then $j : \partial\mathbb{D}^m \hookrightarrow \widehat{M}$ induces a long exact sequence

$$\cdots \to H_{k+1}(\widehat{M}, j(\partial\mathbb{D}^m); \mathbb{Z}) \to H_k(j(\partial\mathbb{D}^m); \mathbb{Z}) \to H_k(\widehat{M}; \mathbb{Z}) \to H_k(\widehat{M}, j(\partial\mathbb{D}^m); \mathbb{Z}) \to \cdots$$

But note that $M \sim \widehat{M}/j(\partial\mathbb{D}^m)$, so

$$H_\bullet(\widehat{M}, j(\partial\mathbb{D}^m); \mathbb{Z}) = H_\bullet(\widehat{M}/j(\partial\mathbb{D}^m; \mathbb{Z}) = H_\bullet(M; \mathbb{Z}).$$

By hypothesis, $H_\bullet(M; \mathbb{Z}) = 0$, so we conclude that $H_\bullet(j_\partial) : H_\bullet(j(\partial\mathbb{D}^m); \mathbb{Z}) \to H_\bullet(\widehat{M}; \mathbb{Z})$ is an isomorphism. Hence by Whitehead's theorem, and the fact that $\pi_1(\widehat{M}) = \{1\}$, we see that $j_\partial : \partial\mathbb{D}^m \hookrightarrow \widehat{M}$ is a homotopy equivalence. Thus $(\widehat{M}; \partial\mathbb{D}^m, \partial M)$ is an h-cobordism.

By Smale's theorem, there is an equivalence

$$F : \widehat{M} \xrightarrow{\simeq} \partial\mathbb{D}^m \times [0,1]; \partial\mathbb{D}^m \times \{0\}, \partial\mathbb{D}^m \times \{1\}).$$

But M is clearly recovered as

$$\begin{tikzcd} \partial\mathbb{D}^m \arrow[r, "j_\partial"] \arrow[d, hook] & \widehat{M} \arrow[d, dashed] \\ \mathbb{D}^m \arrow[r, dashed] & M \end{tikzcd}$$

so $M \simeq (\mathbb{S}^{m-1} \times [0,1]) \cup_{\mathbb{S}^{m-1}} \mathbb{D}^m \simeq \mathbb{D}^m$. □

Corollary 2 (Poincaré conjecture in high dimensions)**.** M^m *homotopy sphere,* $m \geqslant 6 \Longrightarrow M$ *is homeomorphic to* $\mathbb{S}^m$*.*

Proof. As before, start with an embedding $j : \mathbb{D}^m \hookrightarrow M^m$, and let $\widehat{M} := M \diagdown j(\overset{\circ}{\mathbb{D}}{}^m)$, so that $H_k(\widehat{M}, j(\mathbb{D}^m); \mathbb{Z}) = H_k(M; \mathbb{Z})$ still holds. The long exact sequence of the pair $(\widehat{M}, j(\partial\mathbb{D}^m))$ implies that

$$H_k(j(\mathbb{D}^m); \mathbb{Z}) = H_k(\widehat{M}; \mathbb{Z}), \quad k \leqslant n-2,$$

since $H_k(M; \mathbb{Z}) = H_k(\mathbb{S}^m; \mathbb{Z}) = 0$ for $k \neq 0, m$. For the case $k = m-1$ we have

$$0 \to H_m(M); \mathbb{Z}) \to H_{m-1}(j(\mathbb{D}^m); \mathbb{Z}) \to H_{m-1}(\widehat{M}; \mathbb{Z}) \to 0;$$

[2]Note that this is the homotopy-theoretic version of (1) above.

but note that maps the fundamental class of M to that of $j(\partial\mathbb{D}^m)$:

$$H_m(M);\mathbb{Z}) \ni [M] \mapsto [j(\mathbb{D}^m)] \in H_{m-1}(j(\mathbb{D}^m);\mathbb{Z}),$$

and thus $H_{m-1}(\widehat{M};\mathbb{Z}) = 0$.

Hence $\pi_1(\partial\widehat{M}) = \{1\}$, $\quad H_\bullet(\widehat{M};\mathbb{Z}) = 0$, so by Corollary 1, there is a diffeomorphism $i : \mathbb{D}^m \xrightarrow{\simeq} \widehat{M}$.

Consider the diffeomorphism $f := j \circ i^{-1} : i(\partial\mathbb{D}^m) \xrightarrow{\simeq} j(\partial\mathbb{D}^m)$. In general, it is *not* possible to extend f to a diffeomorphism

$$\widetilde{f} : i(\mathbb{D}^m) \xrightarrow{\simeq} j(\mathbb{D}^m);$$

however, we can extend f to a *homeomorphism* $\widetilde{f} : i(\mathbb{D}^m) \xrightarrow{\sim} j(\mathbb{D}^m)$ by the so-called 'Alexander trick' :

$$\widetilde{f} : i(x) \mapsto |x| f\left(\frac{x}{|x|}\right).$$

We can now define a homeomorphism $F : \mathbb{S}^m \to M^m$ by

$$\begin{tikzcd}
\mathbb{D}^m_- \arrow[r, hook] \arrow[dr, "\widetilde{f}\circ i"'] & \mathbb{S}^m \arrow[d, dashed, "F", "\sim"'] & \mathbb{D}^m_+ \arrow[l, hook'] \arrow[dl, "j"] \\
& M &
\end{tikzcd}$$

□

1.2. **Exercises.**

(1) Recall the definition of the weak and strong topologies in the function spaces $C^k(M, M')$, and that $C^r_W(M, M')$ has a complete metric.
(2) Show that $\mathsf{Prop}(M, M') \subset C^0_S(M, M')$ is a connected component.
(3) Let $U \subset M$ be open. The restriction map $C^r(M, M') \to C^r(U, M')$, $0 \leqslant r \leqslant \infty$, is continuous for the weak topology, but not always the strong. However, it is an open map for the strong topologies, and not always for the weak topology.
(4) A submanifold $X \subset W$ of a manifold with boundary is called **neat** if $\partial X = X \cap \partial W$, and X is not tangent to ∂W at any point $x \in \partial X$. Show that if $y \in M$ is a regular value for $f : W \to M$ and $f|_{\partial W} : \partial W \to M$, then $f^{-1}(y) \subset W$ is a neat submanifold.
(5) If $f : M \longrightarrow M'$ is smooth, and $X \subset M'$ is a submanifold, we say that f is **transverse** to X, written $f \pitchfork X$, if
$$\operatorname{im} d_x f + T_{f(x)}X = T_{f(x)}M', \quad x \in f^{-1}(X).$$
Suppose now that W is a manifold with boundary, and $f : W \to M'$ is smooth. If $f, f|_{\partial W} \pitchfork X$, then $f^{-1}X \subset W$ is a neat submanifold, and $\operatorname{codim}(f^{-1}X \subset W) = \operatorname{codim}(X \subset M')$.
(6) Every closed subpace $X \subset M$ can be described as $X = f^{-1}(0)$, where $f : M \to \mathbb{R}$ is a smooth function.
(7) Can you find a smooth $f : \mathbb{T}^2 \to \mathbb{R}$ with exactly three critical points ?
(8) Show that if W if a compact manifold with boundary, there can be no continuous map $r : W \to \partial W$ extending $\mathrm{id}_{\partial W}$.

2. Lecture Two. Normal Forms of smooth maps. Morse functions.

A general goal of this course is to understand how to extract information about the topology of M by means 'good' functions $f : M \to \mathbb{R}$.

We will be mostly concerned with *compact* manifolds without boundary, but many natural constructions lead us away from this more manageable case. When M is non-compact, we will typically demand that $f : M \to \mathbb{R}$ be **proper**.

Aside : proper maps

Recall that a map $f : M \to M'$ is said to be proper if f^{-1} takes compact sets to compact sets. The subspace $\mathsf{Prop}^k(M, M') \subset C^k(M, M')$ of proper maps is open in the strong C^k-topology, for every $k \geqslant 0$.

Exercise : if f is proper, then $f\mathsf{Crit}(f) \subset M'$ is closed.

One very strong reason to deal exclusively with proper maps is that non-proper maps may not reflect any of the topology of M. As an illustration, let us convene that an **open manifold** is a manifold, none of whose connected components is compact without boundary. Then we have

Theorem 5 (Gromov)**.** *On every open manifold M, there is $f \in C^\infty(M)$ with $\mathsf{Crit}(f) = \varnothing$.*

The catch is that such f cannot be proper.

Let us go back to our $f : M \to \mathbb{R}$ (proper if M is non-compact). We inaugurate the notation

$$M_t := f^{-1}(-\infty, t] \subset M.$$

Exercise : this is a smooth manifold with boundary $\partial M_t = f^{-1}(t)$ when t is a regular value for f.

Assume now that $[a, b] \subset \mathbb{R}$ consists only of regular values for f. Our first goal in this lecture is to prove the

Theorem 6 (Structure Theorem I)**.** *M_b is diffeomorphic to M_a, and the inclusion $M_a \hookrightarrow M_b$ is a strong deformation retraction.*

Before we give the proof, a short reminder comes in handy.

Aside : Vector fields and their flows

Recall that, by the Fundamental Theorem of ODEs, a vector field $w \in \mathfrak{X}(M)$ defines a **local flow**. That is, there is

$$\phi : M \times \mathbb{R} \supset \mathrm{dom}(\phi) \longrightarrow M,$$

where $\mathrm{dom}(\phi)$ is an open containing $M \times \{0\}$, with the property that, for each $x \in M$, $c(t) := \phi^t(x)$ is the maximal trajectory of w with initial condition $c(0) = x$. Being a trajectory of w means that $\frac{dc}{dt} = w \circ c$; by 'maximal trajectory' we mean that

$$c : \mathrm{dom}(\phi) \cap \{x\} \times \mathbb{R} =: (a_x, b_x) \to M$$

cannot be extended any further.

Note that $\phi^s(\phi^x(x)) = \phi^{t+s}(x)$ whenever either side of the equation is defined.

When $\mathrm{dom}(\phi) = M \times \mathbb{R}$, we say that ϕ is the **flow** of w; in this case, ϕ determines a group homomorphism $\phi : (\mathbb{R}, +) \to (\mathrm{Diff}(M), \circ)$, and we will say that w is **complete**. **Exercise :** w is complete if it is compactly supported.

However,

Example 4. *Neither* $\partial/\partial t \in \mathfrak{X}(\mathbb{R}\setminus 0)$ *nor* $(1+t^2)\partial/\partial t \in \mathfrak{X}(\mathbb{R})$ *are complete*[3].

There is a classical condition to be imposed on w to ensure that it give rise to a flow.

Definition 4. *A Riemannian metric g on a manifold M is called* **complete** *if the geodesics of its Levi-Civita connection are defined at all times.*

Complete Riemannian metrics exist on all manifolds of finite dimension.

Definition 5. *A vector field $w \in \mathfrak{X}(M)$ is said to have* **bounded velocity** *if there exists a* complete *Riemannian metric g on M, for which $\|w\|$ is bounded by some real number K :*

$$\sup_{x\in M} \|w_x\| \leqslant K < +\infty.$$

Lemma 1. *Let (M, g) be complete.*

(1) $(a, b) \subset \mathbb{R}$ a bounded interval, and $c : (a, b) \to M$ a curve of finite length :

$$\int_a^b \|c'(t)\|dt < \infty.$$

Then $\operatorname{im} c \subset M$ *is precompact.*

(2) Suppose $c(t)$ is a maximal trajectory of $w \in \mathfrak{X}(M)$, $c : J \to M$, where $J \subset \mathbb{R}$ is an interval containing 0. Then :
- $[0, +\infty) \not\subseteq J \Longrightarrow \int_0^b \|c'(t)\|dt = \infty$;
- $(-\infty, 0] \not\subseteq J \Longrightarrow \int_a^0 \|c'(t)\|dt = \infty$;

Proof. **(1) :** It suffices[4] to show that, for every $\varepsilon > 0$, there exist $x_0, ..., x_N \in \operatorname{Cl}\operatorname{im} c$ such that the ε-balls around x_i cover it : $\cup_1^N B_\varepsilon(x_i) \supset \operatorname{Cl}\operatorname{im} c$. But

$$\int_a^b \|c'(t)\|dt < \infty \quad \Longrightarrow \quad \exists a = t_0 < t_1 < \cdots < t_N = b, \quad \int_{t_i}^{t_{i+1}} \|c'(t)\|dt < \varepsilon,$$

so

$$\operatorname{Cl}(\operatorname{im} c) \subset \bigcup_0^N B_\varepsilon(c(t_i)).$$

(2) : If c is maximal, and $[0, \infty) \not\subseteq J$, then $c(t)$ has no limit point as $t \to b$, $b := \sup\{t : t \in J\} < \infty$. It then follows from the first part of the lemma that $\int_0^b \|c'(t)\|dt = \infty$. The other case is completely analogous. $\square$

Definition 6. *An* **isotopy** *ψ of a smooth manifold M is a smooth map*

$$\psi : M \times J \to M,$$

where $J \subset \mathbb{R}$ is an interval containing 0, each $\psi_t := \psi(\cdot, t) : M \to M$ is a diffeomorphism, and $\psi_0 = \mathrm{id}_M$.

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[3]For the second example, note that a solution curve $c(t)$ to $(1 + t^2)\partial/\partial t$ with initial condition $c(0) = 0$ is $c(t) = \tan t$, which cannot be extended beyond $(-\pi/2, +\pi/2)$.

[4]A metric space (X, d) is called **totally bounded** if, for every $\varepsilon > 0$, X can be covered by finitely many ε-balls. A complete metric space is compact iff it is totally bounded. Indeed, it is clear that any compact space is totally bounded. On the other hand, if a space is totally bounded, to show that it is compact it is enough to show that every sequence $(x_n)_{n\geqslant 0}$ has a Cauchy subsequence $(x_{n_k})_{k\geqslant 0}$. Cover X with finitely many balls $B_1, ..., B_N$ of radius 1; then one of the balls, say B_1, must contain infinitely many terms of (x_n). This defines a subsequence $s_1 \subset (x_n)$, and the distance between any two points in s_1 is no greater than 1. Now cover B_1 by finitely many balls of radius 1/2; again, we can select a subsequence $s_2 \subset s_1$ of points lying in one single 1/2-ball. Inductively, we define then a sequence of subsequences $(x_n) \supset \cdots \supset s_k \supset s_{k+1} \supset \cdots$, with each s_k lying in a ball of radius $1/k$; hence a sequence $x_{n_k} \in s_k \setminus s_{k_1}$ must be Cauchy.

The flow of a complete vector field is an example of an isotopy.

An isotopy ψ gives rise to a **time-dependent vector field**, i.e., a one-parameter family $t \mapsto w_t$ of vector fields on M, defined by

$$\frac{d\psi_t}{dt} = w_t \circ \psi_t, \quad t \in J.$$

Remark 2. *A time-dependent vector field w_t can of course be regarded as an autonomous vector field $\widetilde{w}$ on $M \times J$, to which the above discussion applies to produce a local flow*

$$\widetilde{\phi} : (M \times J) \times \mathbb{R} \supset \mathrm{dom}(\widetilde{\phi}) \to M \times J.$$

Note however that $\widetilde{\phi}$ gives rise to an isotopy of $M \times J$, not of M alone. This can be remedied as follows : consider the autonomous

$$\widehat{w} := w_t + \partial/\partial t \in \mathfrak{X}(M \times J)$$

Let us assume for simplicity that $\widehat{w}$ is complete, and let us denote its flow by $\widehat{\phi}$: $(M \times J) \times \mathbb{R} \to M \times J$. Then $\widehat{\phi}$ satisfies

$$\widehat{\phi}_s(x,t) \in M \times \{t+s\},$$

so $\widehat{\phi}_{r+s}(x,t) = \widehat{\phi}_r \circ \widehat{\phi}_s(x,t)$ wherever this makes sense; in particular, $s \mapsto \mathrm{pr}_M \circ \widehat{\phi}_s$ gives rise to an isotopy of M.

We conclude that :

Lemma 2. *Time-dependent vector fields of bounded velocity give rise to isotopies.*

Proof. □

We conclude this aside by recalling a very useful formula from Calculus. Suppose ψ is an isotopy of M with corresponding time-dependent vector field $w_t \in \mathfrak{X}(M)$. Suppose $t \mapsto \eta_t$ is a time-dependent section of some tensor bundle $E := (\bigwedge^p TM) \otimes (\bigwedge^q T^*M)$.

Lemma 3. $\frac{d}{dt}(\psi_t^* \eta_t) = \psi_t^* \left(L(w_t)\eta_t + \frac{d\eta_t}{dt} \right)$.

Exercise : Prove the Lemma.

Proof of Structure Theorem I. Suppose

$$f\mathsf{Crit}(f) \cap [a,b] = \varnothing.$$

Then $f\mathsf{Crit}(f) \cap [a-\varepsilon, b+\varepsilon] = \varnothing$ for small enough $\varepsilon > 0$. Choose

$$\varrho : [a-\varepsilon, b+\varepsilon] \to [0,1]$$

such that

$$\varrho(t) = \begin{cases} 1 & \text{if } t \in [a-\varepsilon/3, b+\varepsilon/3]; \\ 0 & \text{if } t \notin [a-2\varepsilon/3, b+2\varepsilon/3]. \end{cases}$$

Let

$$w := \frac{-(\varrho \circ f)}{\|\nabla f\|^2} \nabla f \in \mathfrak{X}(M),$$

where $\|\cdot\|$ refers to some auxiliary (complete) Riemannian metric g on M and ∇f denotes the vector field defined by $g(\nabla f, v) = df(v)$.

Observe that

$$(L(w)f)(x) = \begin{cases} -1 & \text{if } f(x) \in [a-\varepsilon/3, b+\varepsilon/3]; \\ 0 & \text{if } f(x) \notin [a-2\varepsilon/3, b+2\varepsilon/3]. \end{cases}$$

f being proper, w is compactly supported, and so gives rise to a flow

$$\phi : M_b \times \mathbb{R} \longrightarrow M_b, \quad \phi_t(M_b) \subset M_{b-t}.$$

In particular, we have a diffeomorphism

$$\phi_{b-a} : M_b \xrightarrow{\simeq} M_a,$$

and

$$\phi|_{M_b \times [0,b-a]} : M_b \times [0, b-a] \longrightarrow M_b$$

is a strong deformation retraction of M_b onto M_a. □

This idea that 'in the absence of critical points we can push down M_t' can be turned around to *detect* critical points of a $f \in C^\infty(M)$.

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Aside : Palais-Smale Condition C

Fix a complete Riemannian manifold (M, g), and let $f : M \to \mathbb{R}$ be given.

Definition 7. *We say that f satisfies* **Condition C** *if, whenever a sequence $(x_n)_{n\geqslant 0}$ in M is such that*

- *$(|f(x_n)|)_{n\geqslant 0} \subset \mathbb{R}$ is bounded, and*
- *$\|\nabla f(x_n)\| \to 0$ as $n \to \infty$,*

then there is a subsequence $(x_{n_k})_{k\geq 0})$ converging in M.

Observe that any proper f satisfies Condition C automatically.

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Lemma 4. *Suppose f is bounded below, and f sastisfies Condition C. Then the flow ϕ^t of $-\nabla f$ is defined for all positive times, and for every $x \in M$, $\lim_{t\to+\infty} \phi^t(x)$ exists and is a critical point of f.*

Proof. Let $B := \inf_{x\in M} f(x) > -\infty$, and consider the maximal trajectory

$$c(t) := \phi^t(x), \quad c : J \to M;$$

we wish to show that $[0, +\infty) \subset J$.

First define $F : (a, b) \to \mathbb{R}$ by $F(t) := f(c(t))$. Then

$$B \leqslant F(t) = F(0) + \int_0^t F'(s)ds = F(0) - \int_0^t \|\nabla f(c(s))\|^2 ds$$
$$\implies \int_0^t \|\nabla f(c(s))\|^2 ds \leqslant F(0) - B.$$

Since the RHS is independent of t, we conclude that

$$\int_0^b \|\nabla f(c(s))\|^2 ds \leqslant F(0) - B.$$

Let us argue by contradiction, and assume that b were finite. By Schwarz's inequality,

$$\int_0^b \|\nabla f(c(s))\| ds \leqslant \sqrt{\int_0^b ds}\sqrt{\int_0^b \|\nabla f(c(s))\|^2 ds} \leqslant \sqrt{b(F(0) - B)}.$$

This implies that $\int_0^b \|\nabla f(c(s))\| ds < +\infty$. But by Lemma 1, $b < +\infty$ implies that $\int_0^b \|\nabla f(c(s))\| ds$ is infinite; the contradiction shows that $b = +\infty$.

But then

$$\int_0^\infty \|\nabla f(c(s))\|^2 ds \leqslant F(0) - B \quad \Longrightarrow \quad \|\nabla f_{c(t)}\|^2 \to 0, \text{ as } t \to \infty,$$

so $\|\nabla f_{c(t)}\| \to 0$. By Condition C, we can find $(t_n)_{n \geqslant 0}$ with $c(t_n) \to x \in M$; by continuity of df, we have

$$x \in \mathsf{Crit}(f).$$

□

We will return to this sort of argument in more detail when we deal with Min-Max theory.

2.1. **Normal Forms.** Having dealt with the regular case, we wish to understand the behavior of f around its *singular* points $x \in \mathsf{Crit}(f)$. Ideally, we should be able to provide a *model* for f around each critical point, depending only on the value of a (a priori known) finite number of derivatives of f at x.

For too badly behaved f, this is way too ambitious.

Example 5. *The maps $f_0, f_1 : \mathbb{R} \to \mathbb{R}$, $f_0(t) = 0$, and*

$$f_1(t) = \begin{cases} e^{-1/t} & \text{if } t \geqslant 0, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

both have 0 as a critical point, and their derivatives at 0 vanish to infinite order, and they behave quite differently at zero.

To weed out such behavior, and still hope to model the singularities of f, we should impose some *non-degeneracy* condition on the critical points $x \in \mathsf{Crit}(f)$.

℧℧

Aside : Germs

Recall that if M, M' are smooth manifolds, and $X \subset M$ is any subspace, we denote by

$$C^\infty(M, M')_X = \{[U, f] : X \subset U \subset M \text{ open}, \, f \in C^\infty(U, M')\},$$

where $[U, f]$ denotes the **germ** of f along X :

$$[U, f] = [U', f'] \quad \Longleftrightarrow \quad \exists U'' \subset U \cap U', \quad f|_{U''} = f'|_{U''}.$$

Two germs $[U, f], [U', f'] \in C^\infty(M, M')_X$ will be called **equivalent** if there exist $U'' \subset U \cap U'$, $V \supset f(X)$ opens, and embeddings $j : U'' \hookrightarrow U$ and $i : V \hookrightarrow M'$, with

$$if|_{U''} = f'|_{U''}j.$$

An equivalence class of germs around X formalizes the notion of 'behavior' around X : two maps $f, f' \in C^\infty(M, M')$ have **the same behavior** around $X \subset M$ iff their germs along X are equivalent.

We will typically be lazy, and write $[f]$ (or just f) instead of $[U, f]$.

We will mostly be concerned with $\mathcal{E} := \mathbb{C}^\infty(\mathbb{R}^m, \mathbb{R})_0$, the set of germs of real functions around zero. Note that this is a *ring*, with the operations

$$[f] + [f'] := [f + f'], \quad [f] \cdot [f'] := [ff'],$$

with additive and multiplicative units $[0]$ and $[1]$ respectively. Observe that $\mathcal{E}$ comes equipped with a natural surjective ring homomorphism

$$\mathsf{ev} : \mathcal{E} \to \mathbb{R}, [f] \mapsto f(0).$$

Since $\mathcal{E}/\mathbb{R}$ is a field, $\mathfrak{m} := \ker(\mathsf{ev})$ is a maximal ideal in $\mathcal{E}$; observe that $[f] \notin \mathfrak{m}$ implies that $[f]^{-1} = [f^{-1}] \in \mathcal{E}$, so $\mathfrak{m} \lhd \mathcal{E}$ is the *unique* maximal ideal – that is, $\mathcal{E}$ is a local ring.

Observe that $[f] \in \mathfrak{m}$ iff

$$f(x) = \int_0^1 \frac{d}{dt}(f(tx))dt = \sum_1^m \left(\int_0^1 \frac{\partial f}{\partial x_i}(tx) \right) \cdot x_i,$$

so $\mathfrak{m} = \sum_1^m \mathcal{E} \cdot x_i$; in particular, $\mathfrak{m}^2 = \sum_1^m \mathcal{E} \cdot x_i x_j$ and thus $[f] \in \mathfrak{m}^2$ iff $0 \in \mathsf{Crit}(f)$. This implies that

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} T_0^*\mathbb{R}^m, \quad [f] + \mathfrak{m}^2 \mapsto d_0 f,$$

is an isomorphism of $\mathcal{E}$-modules.

This observation can be expanded by observing that ev extends to a ring homomorphism

$$\mathsf{Tayl} : \mathcal{E} \to \mathbb{R}[[x_1, ..., x_m]], \quad [f] \mapsto \mathsf{Tayl}(f) := \sum_\alpha \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x_\alpha} x^\alpha,$$

where for a multi-index $\alpha = (\alpha_1, ..., \alpha_m)$, $\alpha_i \geqslant 0$, we set

$$|\alpha| := \sum \alpha_i, \quad \alpha! := \prod_1^m \alpha_i!, \quad x^\alpha = \prod_1^m x_i^{\alpha_i}, \quad \frac{\partial^{|\alpha|}}{\partial x_\alpha} := \prod_1^m \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}.$$

The homogeneous part of degree k of $\mathsf{Tayl}(f)$, denoted by $\mathsf{Tayl}^k(f)$, can be described in a slightly less coordinate-dependent fashion. Indeed, if $f : \mathbb{R}^m \to \mathbb{R}$ is a smooth map, then df can be regarded as a smooth map $df : \mathbb{R}^m \to \mathsf{Hom}(\mathbb{R}^m, \mathbb{R}) \simeq \mathbb{R}^m$, and as such we can take $d(df) := d^2 f : \mathbb{R}^m \to \mathsf{Hom}(\mathbb{R}^m, \mathsf{Hom}(\mathbb{R}^m, \mathbb{R}))$. But recall from Calculus that $d^2 f$ lands inside $\mathsf{Hom}^2(\mathbb{R}^m, \mathbb{R})$, i.e., $d^2 f(v, w)$ is symmetric in its arguments $v, w \in T_0\mathbb{R}^m$. More generally, we denote by $d^k f$ the map $d(d^{k-1} f) : \mathbb{R}^m \to \mathsf{Hom}^k(\mathbb{R}^m, \mathbb{R})$; in this notation,

$$\mathsf{Tayl}^k(f) = \frac{1}{k!} d^k f.$$

Lemma 5. *Let* $[f] \in \mathfrak{m}^2$*. Then*

$$d_0^2 f(v, w) = [\widetilde{v}, [\widetilde{w}, f]](0) = [\widetilde{w}, [\widetilde{v}, f]](0),$$

where $\widetilde{v}, \widetilde{w}$ *are any two germs of vector fields around zero extending* $v, w \in T_0\mathbb{R}^m$*, respectively.*

Proof. Note that

$$[\widetilde{v}, [\widetilde{w}, f]] - [\widetilde{w}, [\widetilde{v}, f]] = [[\widetilde{v}, \widetilde{w}], f](0) = d_0 f([\widetilde{v}, \widetilde{w}]) = 0$$

since $0 \in \mathsf{Crit}(f)$. Hence $[\widetilde{v}, [\widetilde{w}, f]](0) = [\widetilde{w}, [\widetilde{v}, f]](0)$. But the LHS can be expressed as

$$[\widetilde{v}, [\widetilde{w}, f]](0) = d([\widetilde{w}, f])(v),$$

which shows that it is independent of the choice of extension $\widetilde{v}$, whereas

$$[\widetilde{w}, [\widetilde{v}, f]](0) = d([\widetilde{v}, f])(w)$$

shows that it is independent of the extension $\widetilde{w}$. Now express f in coordinates and conclude that the quantity above equals $d_0^2 f(v, w)$ (**exercise**). □

Now recall if $B : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is a symmetric bilinear form, there exist integers $0 \leqslant \lambda, \nu \leqslant m$ and a linear basis $(e_i)_1^m$ of $\mathbb{R}^m$ with

$$B(e_i, e_j) = \begin{cases} -1 & \text{if } i = j \text{ and } i \leqslant \lambda, \\ +1 & \text{if } i = j \text{ and } \lambda < i \leqslant m - \nu, \\ 0 & \text{if } i \neq j \text{ or } i > m - \nu. \end{cases}$$

The integer ν is called the **nullity** of B; the form is called **non-degenerate** if $\nu = 0$. The integer λ, on the other hand, is called the **index** of B. Observe that $\nu, \lambda, (m - \lambda - \nu)$ are the dimensions of the maximal subspaces $W \subset \mathbb{R}^m$ where B restricts to zero, a negative-definite form, and a positive-definite form, respectively.

Lemma 6. *Let $[f] \in \mathfrak{m}^2$, and let $\mathsf{Jac}(f) \lhd \mathcal{E}$ denote the ideal spanned by the partial derivatives $\frac{\partial f}{\partial x_i}$. Then $d_0^2 f$ is non-degenerate only if $\mathsf{Jac}(f) = \mathfrak{m}$.*

Proof. Of course, $[f] \in \mathfrak{m}^2$ implies that $\mathsf{Jac}(f) \subset \mathfrak{m}$, so one inclusion always holds.

Suppose $d_0^2 f$ were non-singular. Then $d_0(df) = d_0^2 f : \mathbb{R}^m \to \mathsf{Hom}(\mathbb{R}^m, \mathbb{R})$ is a linear isomorphism; hence by the Inverse Function Theorem, we can express the coordinates x_i as

$$x_i = x_i(\partial f/\partial x_1, ..., \partial f/\partial x_m) \quad \Longrightarrow x_i = \sum_j a_{ij} \cdot \frac{\partial f}{\partial x_j},$$

for some $a_{ij} \in \mathcal{E}$. Since the x_i's span $\mathfrak{m}$, we have $\mathsf{Jac}(f) = \mathfrak{m}$. $\square$

Having described the local picture, we can transfer our definitions to the manifold setting :

Definition 8. *The* **Hessian** $\mathsf{Hess}_x(f)$ *is the bilinear form*

$$T_xM \times T_xM \to \mathbb{R}, \quad \mathsf{Hess}_x(f)(v, w) := d_x^2 f(v, w),$$

corresponding to the critical point $x \in \mathsf{Crit}(f)$. A critical point $x \in \mathsf{Crit}(f)$ is called **non-degenerate** *if $\mathsf{Hess}_x(f)$ is non-singular. If x is a non-degenerate critical point, its* **index** *$\lambda = \lambda(f, x)$ is*

$$\lambda(f, x) := \max\{\dim W : \mathsf{Hess}_x(f)|_W \text{ is negative-definite.}\}$$

A function $f \in C^\infty(M)$ will be called **Morse** *if all of its critical points are non-degenerate.*

We will write $\mathsf{Morse}(M) \subset C^\infty(M)$ for the subspace of Morse functions.

Lemma 7. *(1) $\mathsf{Morse}(M)$ is open in the strong C^2-topology, $\mathsf{Morse}(M) \subset C^2_S(M)$.*
(2) $\lambda(f, x) + \lambda(-f, x) = \dim M$.

Proof. Immediate. $\square$

We wish to prove now :

Theorem 7 (Morse Lemma)**.** *If $f \in \mathfrak{m}^2 \diagdown \mathfrak{m}^3$, there exists an embedding*

$$\psi : 0 \in U \hookrightarrow \mathbb{R}^m, \quad \psi^* f = \frac{1}{2}\mathsf{Hess}_x(f) \in \mathcal{E},$$

where we regard $\mathsf{Hess}_x(f)$ as a smooth function $\mathsf{Hess}_x(f) : T_xM \to \mathbb{R}$ by the rule $v \mapsto \mathsf{Hess}_x(v, v)$.

We need a technical lemma first.

Lemma 8 (Auxiliary Lemma)**.** *If $f \in \mathfrak{m}^2 \diagdown \mathfrak{m}^3$, and $\delta \in \mathfrak{m}^2$, there exists a time-dependent vector field w_t around zero, $t \in [0, 1]$, for which $[w_t, f + t\delta] = -\delta$ and $w_t(0) = 0$ for all t.*

Proof. Note that $\delta \in \mathfrak{m}^3$ implies that $\nabla\delta \in \mathfrak{m}^2$, so

$$\nabla\delta = B(x)x, \quad B(0) = 0.$$

On the other hand, $\mathsf{Jac}(f) = \mathfrak{m}$, so $x = A(x)\nabla f$. Hence

$$\begin{cases} x = A(x)\left(\nabla(f+t\delta)\right) - tA(x)\nabla\delta \\ \nabla\delta = B(x)x \end{cases} \implies \left(\mathrm{id} + tA(x)B(x)\right)x = A(x)\nabla(f+t\delta).$$

Now, $B(0) = 0$ ensures that

$$x = C_t(x)\nabla(f+t\delta), \quad C_t(x) := \left(\mathrm{id} + tA(x)B(x)\right)^{-1} A(x),$$

which means that each of germs of the coordinate functions x_i can be written as

$$x_i = [v_t^i, f + t\delta]$$

for some germ of time-dependent vector field v_t^i.

Now write $\delta = \sum a_{ij}x_ix_j$ and let

$$w_t := \sum_{i,j} a_{ij}x_jv_t^i;$$

then $[w_t, f+t\delta] = -\delta$ as promised, and $w_t(0) = 0$ for all t. □

Proof of Theorem 7. First observe that

$$f_t := (1-t)f + \frac{t}{2}\mathsf{Hess}(f) = f + t\delta, \quad \delta := \frac{1}{2}\mathsf{Hess}(f) - f, \quad t \in [0,1],$$

defines a smooth family $f_t \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$. Note that $\delta \in \mathfrak{m}^3$.

We seek a germ of isotopy ψ_t around 0, such that $\psi_t(0) = 0$ and

$$\psi_t^* f_t = f, \quad t \in [0,1]$$

The latter condition is equivalent to

$$0 = \frac{d}{dt}(\psi_t^* f_t) \iff L(w_t)f_t + \delta = 0,$$

and the former to $w_t(0) = 0$, where w_t denotes the germ of time-dependent vector field corresponding to ψ_t.

But by the Auxiliary Lemma 8, such w_t exists. □

Definition 9. *If $f \in C^\infty(M)$ and $p \in \mathsf{Crit}(f)$ is non-degenerate, a* **Morse chart** *around p is an embedding $\psi : U \hookrightarrow M$ of an open around $0 \in \mathbb{R}^m$ putting f in normal form :*

$$\psi^* f = Q_{\lambda(f,p)},$$

where $Q_{\lambda(f,p)}$ stands for the standard quadratic form of index $\lambda = \lambda(f,p)$, $Q_{\lambda(f,p)} = -\sum_1^\lambda x_i^2 + \sum_{\lambda+1}^m x_i^2$.

2.2. **Exercises.**

(1) If $f \in \mathsf{Morse}(M)$ and $f' \in \mathsf{Morse}(M')$ $i = 0,1$, then $F := \mathrm{pr}_M^* f + \mathrm{pr}_{M'}^* f' \in \mathsf{Morse}(M \times M')$. Determine the critical points of F and their indices in terms of those of f, f'.
(2) Give an example of isolated and non-isolated *degenerate* critical points.
(3) Show that if $[f] \in \mathfrak{m}\setminus\mathfrak{m}^2$, then f has the same behavior as d_xf.
(4) Show that if $f \in \mathsf{Morse}(M^m)$ and $|\mathsf{Crit}(f)| = 2$, then M is homeomorphic to $\mathbb{S}^m$.
(5) Show that every symmetric bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is equivalent to (exactly) one of the form $-\sum_1^\lambda x_i^2 + \sum_{\lambda+1}^n x_i^2$, $0 \leqslant \lambda \leqslant n$.

3. Lecture Three. Abundance of Morse functions.

3.1. **Thom Transversality Theorem.** Recall that if M, M' are smooth manifolds, we say that $f, f' \in C^\infty(M, M')$ have the same **k-jet** at $x \in M$ iff all the partial derivatives of f and f' at x agree up to order k, in which case we write $j_k f(x) = j_k f'(x)$.

The collection

$$J_k(M, M') := \{j_k f(x) : f \in C^\infty(U, M'), x \in U\}$$

of all k-jets of (partially defined) maps $M \to M'$ has a natural structure of smooth manifold. It comes equipped with **source-** and **target** maps,

$$s : J_k(M, M') \to M, \quad j_k f(x) \mapsto x$$
$$t : J_k(M, M') \to M', \quad j_k f(x) \mapsto f(x);$$

which are fibre bundles, and bundle maps

$$p^k_{k-1} : J_k(M, M') \to J_{k-1}(M, M'), \quad j_k f(x) \mapsto j_{k-1} f(x)$$

so that we have commuting diagrams

$$\begin{tikzcd} & J_k(M,M') \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[dd, "p^k_{k-1}"'] & \\ M & & M' \\ & J_{k-1}(M,M') \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

There is also an assignment

$$j_k : C^\infty(M, M') \to C^\infty(M, J_k(M, M')), \quad f \mapsto [x \mapsto j_k f(x)],$$

which we refer to as the **k-jet map**.

Recall that a subspace A of a topological space X is called **residual** if it is the countable intersection of open, dense subspaces :

$$A = \bigcap_{n \geqslant 0} U_n, \quad U_n \subset X \text{ open and } \operatorname{Cl} U_n = X, \quad \forall n.$$

A topological space X is called **Baire** if every residual subspace is dense.

Theorem 8. *A residual subspace of a complete metric space is dense. Every weakly closed subspace of $C^r_S(M, M')$ is a Baire space.*

Proof. See [9]. □

We can now remind the reader of :

Theorem 9 (Thom Transversality Theorem, v. 1)**.** *If $X \subset J_k(M, M')$ is a submanifold, then the space of $f \in C^r(M, M')$ with $j_k f \pitchfork X$ is residual in $C^r_S(M, M')$ for $r > k$, and is open if X is closed.*

Aside : Multijet bundles

We will make good use of an extension of Thom Transversality, whose setting we describe.

Fix an integer $l > 0$ and consider

$$J_k^{(l)}(M, M') \subset \prod_1^l J_k(M, M'), \quad J_k^{(l)}(M, M') := (\prod_1^k s)^{-1} M^{(l)},$$

where

$$M^l \supset M^{(l)} := \{(x_1, ..., x_l) : i \neq j \Longrightarrow x_i \neq x_j\}.$$

Then clearly $J_k^{(l)}(M, M')$ is a bundle over $M^{(l)}$, with projection

$$s^{(l)} : (j_k f_1(x_1), ..., j_k f_l(x_l)) \mapsto (x_1, ..., x_l),$$

and there is an induced **multijet map**

$$j_k^{(l)} : C^k(M, M') \to C^0(M^{(l)}, J_k^{(l)}(M, M'))$$

$$j_k^{(l)} f : M^{(l)} \ni (x_1, ..., x_l) \mapsto (j_k f(x_1), ..., j_k f(x_l)) \in J_k^{(l)}(M, M').$$

Theorem 10 (Thom Transversality Theorem, v. 2)**.** *If $X \subset J_k^{(l)}(M, M')$ is a submanifold, then the space of $f \in C^r(M, M')$ with $j_k^{(l)} f \pitchfork X$ is residual in $C_S^r(M, M')$ for $r > k$, and is open if X is closed.*

Proof. See [7]. □

Now we put these ideas to use.

Definition 10. *The* **singularity set** $S_1 \subset J_1(M, \mathbb{R})$ *is the subspace defined by*

$$S_1 = \{j_1 f(x) : d_x f = 0\}.$$

Lemma 9. *S_1 is a closed submanifold, of codimension* $\operatorname{codim}(S_1 \subset J_1(M, \mathbb{R})) = \dim M$*.*

$x \in \mathsf{Crit}(f)$ iff $j_1 f(x) \in S_1$. Moreover, x is non-degenerate iff $j_1 f \pitchfork S_1$ at x.

Corollary 3. $\mathsf{Morse}(M) \subset C_S^2(M, \mathbb{R})$ *is open and dense. If $f \in \mathsf{Morse}(M)$, $\mathsf{Crit}(f)$ is discrete.*

Proof. Combine Lemma 9 with Theorem 9 for the first statement. For the second, observe that $\operatorname{codim}(\mathsf{Crit}(f) \subset M) = \dim M$, so $\mathsf{Crit}(f)$ is a zero-dimensional submanifold. □

Definition 11. *A Morse function $f \in \mathsf{Morse}(M)$ is called* **resonant** *if there exist distinct critical points $x, y \in \mathsf{Crit}(f)$ at the same critical value : $f(x) = f(y)$. Otherwise it is called* **non-resonant***, and the space of all such will be written* $\mathsf{Morse}_{\neq}(M)$*.*

Lemma 10. $\mathsf{Morse}_{\neq}(M) \subset C_S^2(M, \mathbb{R})$ *is open and dense.*

Proof. First observe that $\mathsf{Morse}_{\neq}(M) \subset C_S^2(M, \mathbb{R})$ is clearly open, so we need only show that it contains a dense subspace.

Consider the multijet bundle $J_1^{(2)}(M, \mathbb{R}) \to M^{(2)} = M \times M \setminus \Delta_M$, and let $S_{\neq} \subset J_1^{(2)}(M, \mathbb{R})$ be the subspace defined by

$$S_{\neq} := (S_1 \times S_1) \cap (t \times t)^{-1}(\Delta_{\mathbb{R}}).$$

One readily sees that $S_{\neq}$ is a submanifold of codimension $2 \dim M + 1$, and hence $j_1^{(2)} f \pitchfork S_{\neq}$ at (x_1, x_2) means $j_1^{(2)} f(x_1, x_2) \notin S_{\neq}$.

By Theorem 10, the subspace $U \subset C_S^2(M, \mathbb{R})$ of those f with $j_1^{(2)} f \pitchfork S_{\neq}$ is open and dense; thus $\mathsf{Morse}(M) \cap U$ is open and dense. But $j_1^{(2)} f$ maps (x_1, x_2) into $S_{\neq}$ iff $d_{x_1} f = 0 = d_{x_2} f$ and $f(x_1) = f(x_2)$, so $\mathsf{Morse}_{\neq}(M) \supset \mathsf{Morse}(M) \cap U$. □

3.2. **Concatenating and Factorizing Cobordisms.**

In view of Lemma 10, any $f \in C^\infty(M)$ can be perturbed ever so slightly to a non-resonant Morse function.

Suppose M is compact, so that $\mathsf{Crit}(f)$ is *finite*. Order the critical values

$$\{c_1 < c_2 < \cdot < c_N\} = f\mathsf{Crit}(f),$$

and let $-\infty = a_0 < a_1 < \cdots a_{N-1} < a_N = +\infty$, with $c_i \in (a_{i-1}, a_i)$ for every $1 \leqslant i \leqslant N$.

Then

$$\mathcal{C}_i := \left(W_i, f^{-1}(a_{i-1}), f^{-1}(a_i)\right), \quad W_i := f^{-1}[a_{i-1}, a_i].$$

are cobordisms, and $M = \cup_i W_i$. Note that $f \pitchfork a_i$ for every $0 < i < N$, and $f_i := f|_{W_i}$ contains a single critical point. We give this situation a special name :

Definition 12. *A cobordism $\mathcal{C} = (W; M_0, M_1)$ is called* **elementary** *if there exists a smooth function $f : W \to [a, b]$, with $f \pitchfork \partial[a, b]$, $f^{-1}(a) = \partial_0 W$, $f^{-1}(b) = \partial_1 W$, and $\mathsf{Crit}(f) = \{p\}$, with $a < f(p) < b$.*

Definition 13. *Let W be a manifold with boundary $\partial W \hookrightarrow W$. By a* **distinguished submanifold** *$X \subset W$ we will refer to either a connected component of the boundary $X \subset \partial W$, or to a cooriented interior submanifold $X \subset (W \smallsetminus \partial W)$.*

A **collar** *of a distinguished submanifold X is an embedding $c : X \times I(X, \varepsilon) \hookrightarrow W$ with $c|_X = \mathrm{id}_X$, and $c_*(\partial/\partial t)$ pointing inwards if $X \subset \partial W$, and in the positive coorientation if $X \subset (W \smallsetminus \partial W)$; here $I(X, \varepsilon) = (-\varepsilon, \varepsilon)$ if X is interior and $I(X, \varepsilon) = [0, \varepsilon)$ if X lies in the boundary.*

Lemma 11 (Collars)**.**

(1) Collars exist.

(2) If $c, c' : X \times I(X, \varepsilon) \hookrightarrow W$ are collars, there is $0 < \delta \leqslant \varepsilon$ and a homotopy of collars $C : X \times I(X, \delta) \times [0, 1] \to W$ joining $c|_{X \times I(X,\delta)}$ to $c'|_{X \times I(X,\delta)}$.

(3) If $C : X \times I(X, \delta) \times [0, 1] \to W$ is a homotopy of collars, there is a collar $\overline{c} : X \times I(X, \delta) \hookrightarrow W$ with

$$\overline{c}|_{X \times I(X,\delta/3)} = C_1|_{X \times I(X,\delta/3)}$$
$$\overline{c}|_{X \times (I(X,\delta) \smallsetminus I(X,2\delta/3))} = C_0|_{X \times (I(X,\delta) \smallsetminus I(X,2\delta/3))}$$

Proof. (1) Using a partition of unity, one constructs on an open $U \subset W$ containing X a vector field $w \in \mathfrak{X}(U)$ with w pointing inwards if $X \subset \partial W$, and w in the positive coorientation if X is interior.

Let $\phi : U \times \mathbb{R} \supset \mathrm{dom}(\phi) \to U$ denote the local flow of w, and choose any embedding $\psi : X \times I(X, \varepsilon) \hookrightarrow \mathrm{dom}(\phi)$ with $\psi|_{X \times \{0\}}$ the inclusion $X \hookrightarrow \mathrm{dom}(\phi)$. Then $c := \phi \circ \psi$ is a collar.

(2) Let $v := c_*(\partial/\partial t), v' := c'_*(\partial/\partial t)$ be defined in a common open $X \subset U$. Define $v_s := (1 - s)v + sv' \in \mathfrak{X}(U)$, for $s \in [0, 1]$, and let

$$\phi_{v_s} : U \times \mathbb{R} \supset \mathrm{dom}(\phi_{v_s}) \longrightarrow U$$

denote the local flow of v_s. Choose a homotopy of embeddings $\psi_s : X \times I(X, \varepsilon) \hookrightarrow \mathrm{dom}(\phi_{v_s})$, $0 \leqslant s \leqslant 1$, with $\psi_s|_X$ the inclusion, and set $C_s := \phi_{v_s} \circ \psi_s : X \times I(X, \varepsilon) \hookrightarrow W$.

(3) Let $s \mapsto w_s$ denote the time-dependent vector field $\frac{dC_s}{ds} \in \mathfrak{X}(\mathrm{im}\, C_s)$, and note that $w_s(x) = 0$ for all $x \in X$ and $s \in [0, 1]$; hence w_s has bounded velocity on some $U'_s \supset X$. Choose then a smooth function $\varrho : X \times I(X, \varepsilon) \times [0, 1] \to \mathbb{R}$, with $\varrho_s = 1$ on a smaller open $U''_s \subset U'_s$ around X, and set $\overline{w}_s := \varrho_s w_s \in \mathfrak{X}(W)$. Then $\overline{w}$ has bounded velocity, and thus generates an isotopy ϕ^s of W with $d_x\phi_s = \mathrm{id}$ for all $x \in X$ and $s \in [0, 1]$, and $\phi^1 C_0$ agrees with C_0 away from X, and with C_1 around it.

□

Corollary 4. *Suppose W, W' are smooth manifolds with boundary, that $X \subset \partial W$ be a sum of* outgoing *connected, and that $h : X \hookrightarrow \partial W'$ embeds X as a sum of* incoming *connected compopnents of $\partial W'$. Then the topological space $W \cup_h W'$ carries a canonical structure of smooth manifold with boundary, and*

$$\partial(W \cup_h W') = (\partial W \diagdown X) \coprod (\partial W' \diagdown h(X))$$

Proof. Suppose for simplicity that X is connected; the general case is argued component-by-component.

We need first introduce a smooth structure on $W \cup_h W'$. Choose collars

$$c : X \times (-\varepsilon, 0] \hookrightarrow W, \quad c' : h(X) \times [0, \varepsilon) \hookrightarrow W'$$

and define the space $W \cup_{h,c} W'$ according to the diagram

$$\begin{tikzcd} X \times ((-\varepsilon, \varepsilon) \diagdown 0) \arrow[r, "H"] \arrow[d] & (W \diagdown X) \coprod (W' \diagdown h(X)) \arrow[d, dashed] \\ X \times (-\varepsilon, \varepsilon) \arrow[r, dashed] & W \cup_{h,c} W' \end{tikzcd}$$

where

$$H(x,t) = \begin{cases} c(x,t) & \text{if } t < 0; \\ c'(h(x),t) & \text{if } t > 0. \end{cases}$$

This exhibits $W \cup_{h,c} W'$ as a *smooth* manifold with the boundary as in the statement.

We need now show that the recipe above is independent of the choices of collars c, c' up to a diffeomorphism.

So suppose γ, γ' are two different choices of collars, and let $W \cup_{h,\gamma} W'$ denote the manifold arising from those choices. Then note that the identity maps $\mathrm{id}_W, \mathrm{id}_{W'}$, glue to a homeomorphism

$$G : W \cup_{h,c} W' \longrightarrow W \cup_{h,\gamma} W'.$$

On the $X \times (-\varepsilon, +\varepsilon)$ part of those manifolds, G reads

$$G = \begin{cases} \gamma c^{-1} & \text{on } \operatorname{im} c; \\ \gamma' c'^{-1} & \text{on } \operatorname{im} c'. \end{cases}$$

According to Lemma 11, c, c' can be modified to a collars $\overline{c}, \overline{c'}$, with

$$\overline{c} = \begin{cases} c & \text{on } X \times (-\varepsilon/3, 0]; \\ \gamma & \text{on } X \times (-\varepsilon, -2\varepsilon/3). \end{cases}, \quad \overline{c'} = \begin{cases} c' & \text{on } h(X) \times [0, \varepsilon/3); \\ \gamma' & \text{on } h(X) \times (2\varepsilon/3, \varepsilon). \end{cases}$$

We then modify G to a *diffeomorphism* $\overline{G} : W \cup_{h,c} W' \xrightarrow{\sim} W \cup_{h,\gamma} W'$,

$$\overline{G} := \begin{cases} G & \text{outside } X \times (-\varepsilon, \varepsilon); \\ \gamma \overline{c}^{-1} & \text{on } \operatorname{im} c; \\ \gamma' \overline{c'}^{-1} & \text{on } \operatorname{im} c'. \end{cases}.$$

□

Definition 14. *We refer to $W \cup_h W'$ as the* **concatenation** *of W, W' along h.*

Example 6. *Let W be any manifold with boundary, and define $2W := W \cup_{\mathrm{id}_{\partial W}} \overline{W}$, the* **double** *of W. Note that $\partial(2W) = \varnothing$.*

Note that, by its very construction, concatenation is 'distributive', in the sense that if we are given a further manifold with boundary W'', $Y \subset \partial W'$ is outgoing, and $h' : Y \hookrightarrow \partial W''$ is an incoming embedding, then there is a natural identification

$$(W \cup_h W') \cup_{h'} W'' \simeq W \cup_h (W' \cup_{h'} W'') .$$

Definition 15. *A* **factorization** *of a manifold with boundary W is a presentation as a concatenation of manifolds with boundary :*

$$W = W_0 \cup_{h_1} W_1 \cup_{h_2} \cdots \cup_{h_k} W_k.$$

Lemma 12. *Every cobordism $\mathcal{C}$ can be factorized into elementary cobordisms.*

Proof. Let $\mathcal{C} = (W; M_0, M_1)$ be a cobordism. Double W to the manifold (without boundary) $2W$, and note that ∂W embeds as a compact submanifold of $2W$.

Choose any $f' : 2W \to [-1, +1]$ with $f' \pitchfork \partial\mathbb{D}^1$ and $f'^{-1}\partial\mathbb{D}^1 = \partial W$. Use Lemma 10 to perturb f' to $f'' \in \mathsf{Morse}_{\neq}(2W)$; choose f'' so C^1-close to f' so that $\partial\mathbb{D}^1$ are regular values for $f'_t := (1-t)f' + tf''$, $0 \leqslant t \leqslant 1$. Then there is a homotopy of embeddings $\psi : \partial W \times [0, 1] \to 2W$ tracking $f_t'^{-1}\partial\mathbb{D}^1$:

$$f'_t \psi_t(\partial_i W) = i, \quad i = 0, 1.$$

By the Isotopy Extension Lemma 13 below, ψ can be extended to an isotopy $\varphi : 2W \times [0, 1] \to 2W$; then

$$f := f'' \circ \varphi_1|_{2W \setminus (\overline{W} \setminus \overline{W})} \in C^\infty(W)$$

is transverse to $\partial\mathbb{D}^1$ and pulls it back to ∂W, and is a non-resonant Morse function in the interior of W. Now choose $a_i \in \mathbb{R} \setminus f\mathsf{Crit}(f)$ such that every $c \in f\mathsf{Crit}(f)$ lies in exactly one interval (a_i, a_{i+1}); then the concatenation of the cobordisms

$$\mathcal{C}_i := \big(W_i, f^{-1}(a_{i-1}), f^{-1}(a_i)\big)$$

is diffeomorphic to W. □

Lemma 13 (Isotopy Extension Lemma)**.** *Let W be a manifold with boundary, and $X \subset W$ a closed submanifold, with either $X \subset (W \setminus \partial W)$ or $X \subset \partial W$. Then every homotopy of embeddings $\psi : X \times [0, 1] \to W$, $\psi_t : X \hookrightarrow W$, whose velocity $\frac{d\psi_t}{dt}$ is bounded, extends to an isotopy $\varphi : W \times [0, 1] \to W$.*

Proof. **Case 1 : $X \subset (W \setminus \partial W)$.**

Consider

$$\widehat{\psi} : X \times [0, 1] \longrightarrow W \times [0, 1], \quad \widehat{\psi}(x, t) = (\psi_t(x), t).$$

The hypotheses ensure that $\widehat{\psi}$ is a closed embedding, and that

$$\widehat{w} := \frac{d\psi_t}{dt} + \partial/\partial t$$

is defined along its image and has bounded velocity.

Choose :

- a tubular neighborhood
$$(W \setminus \partial W) \times I \supset E \xrightarrow{p} \widehat{\psi}(X \times [0, 1]);$$
- a smooth function $\varrho \in C^\infty(E)$, with $\varrho = 1$ around $\widehat{\psi}(X \times [0, 1])$, and whose support meets every fibre of p in a compact set;
- an Ehresmann connection $\mathsf{hor} : \mathfrak{X}(\widehat{\psi}(X \times [0, 1])) \longrightarrow \mathfrak{X}(E)$.

Then set $w := \varrho\mathsf{hor}(\widehat{w}) \in \mathfrak{X}(W \times [0, 1])$ and observe that $w = w_t + f\partial/\partial t$, where $w_t \in \mathfrak{X}(W)$ is supported in the interior of W, and extends $\frac{d\psi_t}{dt}$. Hence w_t gives rise to an isotopy of W extending ψ.

Case 2 : $X \subset \partial W$.

Apply Case 1 twice, first to $X \subset \partial W$, and then to $\partial W \subset W$. □

3.3. **Exercises.**

(1) Show that $J_k(M, M')$ is indeed a smooth manifold, and compute its dimension.
(2) Show that $j_k : C^k(M, M') \to C^0(M, J_k(M, M'))$ is continuous in both the weak and the strong topologies, and has closed image in the weak topology.
(3) Let $M \subset \mathbb{R}^N$ be a submanifold. For each $y \in \mathbb{R}^N$, let $f_y : M \to \mathbb{R}$ denote $x \mapsto \|y - x\|^2$. Show that for y *generic*, $f_y \in \mathsf{Morse}(M)$.(Meaning that the set of points for which the stated property holds is residual).
(4) Compute $\pi_n(\mathbb{S}^m)$ for all $m > n \geqslant 0$.
(5) Two compact manifolds M_0^m, M_1^m are called **(oriented) cobordant** if there exists a (oriented) cobordism $\mathcal{C} = (W; M_0, M_1)$. Show that :
   (a) Being (oriented) cobordant to is an equivalence relation.
   (b) The sets $\mathcal{N}_m$, Ω_m of equivalence classes under cobordism and oriented cobordism relations, respectively, are *abelian groups* under disjoint union $\coprod$.
   (c) If $f, f' : M \to M'$ are homotopic, and transverse to a closed submanifold $X \subset M'$, then $f^{-1}X$ and $f'^{-1}X$ are cobordant. If M, M' and X are orientable, $f^{-1}X$ and $f'^{-1}X$ are oriented cobordant.
   (d) Compute $\mathcal{N}_i$ and Ω_i, for $i = 0, 1$.
(6) Let M, M' be compact smooth manifolds, and let $G := \mathrm{Diff}(M') \times \mathrm{Diff}(M)$ act on $C^\infty(M, M')$ by
$$(\psi, \varphi) : f \mapsto \psi \circ f \circ \varphi^{-1},$$
where G is endowed with the C^∞ topology. A map f is called **stable** if every f' close enough to f lies in the same orbit as f.
   Show that $f \in C^\infty(M, \mathbb{R})$ is stable only if $f \in \mathsf{Morse}_{\neq}(M)$.[5]

[5]We will see later that $\mathsf{Morse}_{\neq}(M)$ is precisely the space of stable functions on M.

4. Lecture Four. Passing a critical level set

4.1. **Surgery.** For every $1 \leqslant \lambda < m$, consider the "standard" diffeomorphisms

$$\mathsf{std}_\lambda : \mathbb{S}^{\lambda-1} \times (\mathring{\mathbb{D}}^{m-\lambda+1} \setminus 0) \xrightarrow{\simeq} (\mathring{\mathbb{D}}^{\lambda} \setminus 0) \times \mathbb{S}^{m-\lambda}$$
$$\mathsf{std}_\lambda : (u, \theta v) \mapsto (\theta u, v), \quad (u,v) \in \mathbb{S}^{\lambda-1} \times \mathbb{S}^{m-\lambda}, \quad \theta \in (0,1).$$

Fix an embedding

$$\varphi : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \hookrightarrow M^m,$$

and consider the *smooth* manifold $\mathsf{Surg}(M, \varphi)$ defined by the pushout diagram

$$\begin{tikzcd}
\mathbb{S}^{\lambda-1} \times (\mathring{\mathbb{D}}^{m-\lambda+1} \setminus 0) \arrow[r, "\varphi"] \arrow[d, "\mathsf{std}_\lambda"'] & M \setminus \varphi(\mathbb{S}^{\lambda-1}) \arrow[d, dashed] \\
\mathring{\mathbb{D}}^{\lambda} \times \mathbb{S}^{m-\lambda} \arrow[r, dashed] & \mathsf{Surg}(M, \varphi).
\end{tikzcd}$$

Observe that $\mathsf{Surg}(M, \varphi)$ comes equipped with a canonical embedding $\mathsf{Surg}(\varphi) : \mathring{\mathbb{D}}^{\lambda} \times \mathbb{S}^{m-\lambda} \hookrightarrow \mathsf{Surg}(M, \varphi)$. Producing $\mathsf{Surg}(M, \varphi)$ out of M has the effect of removing a $(\lambda-1)$-sphere, embedded with trivial normal bundle in M, and replacing it by a $(m-\lambda)$-sphere, also embedded with trivial normal bundle.

Definition 16. *We say that* $\mathsf{Surg}(M, \varphi)$ *is obtained from* M *by a* **surgery of type** λ.

Lemma 14. *If* $\varphi_t : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \hookrightarrow M$ *is a homotopy of embeddings, then* $\mathsf{Surg}(M, \varphi_0) \simeq \mathsf{Surg}(M, \varphi_1)$.

Proof. Extend $\frac{d\varphi_t}{dt} \in \mathfrak{X}(\operatorname{im} \varphi_t)$ to a globally defined (time-dependent) vector field $w_t \in \mathfrak{X}(M)$. We can further demand that the support of w_t be a small neighborhood of $\operatorname{im} \varphi_t$. Denote by ϕ^t the isotopy it generates, and observe that

$$\phi^t(\varphi_t(u, \theta v)) = \varphi_t(u, \theta v).$$

Then

$$\phi^1 \coprod \mathrm{id} : (M \setminus \varphi_0(\mathbb{S}^{\lambda-1}) \coprod \mathring{\mathbb{D}}^{\lambda} \times \mathbb{S}^{m-\lambda} \xrightarrow{\simeq} (M \setminus \varphi_1(\mathbb{S}^{\lambda-1}) \coprod \mathring{\mathbb{D}}^{\lambda} \times \mathbb{S}^{m-\lambda}$$

descends to a diffeomorphism $\mathsf{Surg}(M, \varphi_0) \xrightarrow{\simeq} \mathsf{Surg}(M, \varphi_1)$. $\square$

4.2. **A closer look at model singularities.** Let $L_\lambda \subset \mathbb{R}^\lambda \times R^{m-\lambda+1}$ be the subspace defined by

$$L_\lambda := \{(x, y) : -1 \leqslant Q_\lambda(x, y) \leqslant +1, |x||y| < \sinh 1 \cosh 1\},$$

where as usual Q_λ denotes $Q_\lambda(x, y) = -|x|^2 + |y|^2$.

Note that L_λ is a smooth manifold with boundary $\partial L_\lambda = \partial_{\mathrm{left}} L_\lambda \coprod \partial_{\mathrm{right}} L_\lambda$, where

$$\partial_{\mathrm{left}} L_\lambda := \{(x, y) \in L_\lambda : Q_\lambda(x, y) = -1\}$$
$$\partial_{\mathrm{right}} L_\lambda := \{(x, y) \in L_\lambda : Q_\lambda(x, y) = +1\}.$$

We let $\mathbb{R}_\times$ denote

$$\mathbb{R}_\times := (\mathbb{R}^\lambda \setminus 0) \times (\mathbb{R}^{m-\lambda+1} \setminus 0) \subset \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda+1}.$$

Lemma 15. *There exist diffeomorphisms*

$$\varphi_{left} : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \xrightarrow{\simeq} \partial_{left} L_\lambda$$
$$\varphi_{right} : \mathring{\mathbb{D}}^{\lambda} \times \mathbb{S}^{m-\lambda} \xrightarrow{\simeq} \partial_{right} L_\lambda$$
$$\mathsf{std}^\lambda : \partial_{left} L_\lambda \cap \mathbb{R}_\times \xrightarrow{\simeq} \partial_{right} L_\lambda \cap \mathbb{R}_\times,$$

such that

$$\begin{tikzcd}
\partial_{left} L_\lambda \cap \mathbb{R}_\times \arrow[r, "\varphi\varphi_{left}^{-1}"] \arrow[d, "\mathsf{std}^\lambda"'] & M \setminus \varphi(\mathbb{S}^{\lambda-1}) \arrow[d, dashed] \\
\partial_{right} L_\lambda \arrow[r, dashed] & \mathsf{Surg}(M, \varphi)
\end{tikzcd}$$

$$\begin{tikzcd}
\partial_{right} L_\lambda \cap \mathbb{R}_\times \arrow[r, "\mathsf{Surg}(\varphi)\varphi_{right}^{-1}"] \arrow[d, "\mathsf{std}^\lambda"'] & \mathsf{Surg}(M, \varphi) \setminus \mathsf{Surg}(\varphi)(\mathbb{S}^{m-\lambda}) \arrow[d, dashed] \\
\partial_{left} L_\lambda \arrow[r, dashed] & M
\end{tikzcd}$$

Proof. Define $\mathsf{std}^\lambda : \mathbb{R}_\times \xrightarrow{\simeq} \mathbb{R}_\times$ by the formula

$$\mathsf{std}^\lambda : (x, y) \mapsto \left(\frac{|x|}{|y|} x, \frac{|y|}{|x|} y \right),$$

and observe that std^λ is an *involution*, $\mathsf{std}^\lambda = (\mathsf{std}^\lambda)^{-1}$. Moreover, it induces a diffeomorphism

$$\partial_{\mathrm{left}} L_\lambda \cap \mathbb{R}_\times \xrightarrow{\simeq} \partial_{\mathrm{right}} L_\lambda \cap \mathbb{R}_\times,$$

which we still denote by std^λ.

Now define the diffeomorphisms

$$\varphi_{\mathrm{left}} : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \xrightarrow{\simeq} \partial_{\mathrm{left}} L_\lambda, \quad \varphi_{\mathrm{left}}(u, \theta v) = (u \cosh\theta, v \sinh\theta)$$
$$\varphi_{\mathrm{right}} : \mathring{\mathbb{D}}^{\lambda} \times \mathbb{S}^{m-\lambda} \xrightarrow{\simeq} \partial_{\mathrm{right}} L_\lambda, \quad \varphi_{\mathrm{right}}(\theta u, v) = (u \sinh\theta, v \cosh\theta).$$

Then

$$\begin{tikzcd}
\mathbb{S}^{\lambda-1} \times (\mathring{\mathbb{D}}^{m-\lambda+1} \setminus 0) \arrow[r, "\varphi_{\mathrm{left}}", "\simeq"'] \arrow[d, "\mathsf{std}_\lambda"', "\simeq"] & \partial_{\mathrm{left}} L_\lambda \cap \mathbb{R}_\times \arrow[d, "\simeq"', "\mathsf{std}^\lambda"] \\
(\mathring{\mathbb{D}}^{\lambda} \setminus 0) \times \mathbb{S}^{m-\lambda} \arrow[r, "\simeq", "\varphi_{\mathrm{right}}"'] & \partial_{\mathrm{right}} L_\lambda \cap \mathbb{R}_\times
\end{tikzcd}$$

commutes. Hence

$$\begin{tikzcd}
\partial_{\mathrm{left}} L_\lambda \cap \mathbb{R}_\times \arrow[r, "\varphi\varphi_{\mathrm{left}}^{-1}"] \arrow[d, "\mathsf{std}^\lambda"'] & M \setminus \varphi(\mathbb{S}^{\lambda-1}) \arrow[d, dashed] \\
\partial_{\mathrm{right}} L_\lambda \arrow[r, dashed] & \mathsf{Surg}(M, \varphi).
\end{tikzcd}$$

is also a pushout diagram. On the other hand, the pushout of the outer diagram in

$$\begin{tikzcd}
\mathbb{S}^{\lambda-1} \times (\mathring{\mathbb{D}}^{m-\lambda+1} \setminus 0) \arrow[r, "\varphi_{\mathrm{left}}"] \arrow[d, "\mathsf{std}_\lambda"'] & \partial_{\mathrm{left}} L_\lambda \cap \mathbb{R}_\times \arrow[r, "\varphi\varphi_{\mathrm{left}}^{-1}"] \arrow[d, "\mathsf{std}^\lambda"'] & M \setminus \varphi(\mathbb{S}^{\lambda-1}) \arrow[d] \\
(\mathring{\mathbb{D}}^{\lambda} \setminus 0) \times \mathbb{S}^{m-\lambda} \arrow[r, "\varphi_{\mathrm{right}}"] \arrow[d, "\mathsf{std}_\lambda"'] & \partial_{\mathrm{right}} L_\lambda \cap \mathbb{R}_\times \arrow[r, "\mathsf{Surg}(\varphi)\varphi_{\mathrm{right}}^{-1}"] \arrow[d, "\mathsf{std}^\lambda"'] & \mathsf{Surg}(M,\varphi) \setminus \mathsf{Surg}(\varphi)(\mathbb{S}^{m-\lambda}) \\
\mathbb{S}^{\lambda-1} \times (\mathring{\mathbb{D}}^{m-\lambda+1} \setminus 0) \arrow[r, "\varphi_{\mathrm{right}}"'] & \partial_{\mathrm{left}} L_\lambda \cap \mathbb{R}_\times &
\end{tikzcd}$$

is clearly $M \setminus \varphi(\mathbb{S}^{\lambda-1})$, as the top horizontal arrow equals φ and the left vertical one is identical. Hence

$$\begin{tikzcd}
\partial_{\mathrm{right}} L_\lambda \cap \mathbb{R}_\times \arrow[r, "\mathsf{Surg}(\varphi)\varphi_{\mathrm{right}}^{-1}"] \arrow[d, "\mathsf{std}^\lambda"'] & \mathsf{Surg}(M,\varphi) \setminus \mathsf{Surg}(\varphi)(\mathbb{S}^{m-\lambda}) \arrow[d, dashed] \\
\partial_{\mathrm{left}} L_\lambda \arrow[r, dashed] & M
\end{tikzcd}$$

□

Theorem 11. *There is an elementary cobordism $(\mathcal{C}, f)$ of index λ between M and $\mathsf{Surg}(M,\varphi)$.*

Proof. For every $(x,y) \in L_\lambda$, the curve

$$t \mapsto (tx, t^{-1}y), t > 0,$$

is orthogonal to the level sets $Q_\lambda = c$, $c \neq 0$.

Observe that

$$t = t(x,y) := \sqrt{\frac{1+\sqrt{1+4|x|^2|y|^2}}{2|x|^2}} \Longrightarrow Q_\lambda(tx, t^{-1}y) = -1;$$

hence we obtain a diffeomorphism

$$\psi : L_\lambda \cap \mathbb{R}_\times \xrightarrow{\sim} (\partial_{\mathrm{left}} L_\lambda \cap \mathbb{R}_\times) \times [-1,+1],$$
$$\psi : (x,y) \mapsto \big((t(x,y)x, t(x,y)^{-1}y), Q_\lambda(x,y)\big).$$

We can thus form the smooth manifold W by

$$\begin{tikzcd}
L_\lambda \cap \mathbb{R}_\times \arrow[r, "(\varphi\varphi_{\mathrm{left}}^{-1} \times \mathrm{id})\psi"] \arrow[d] & (M \setminus \varphi(\mathbb{S}^{\lambda-1}) \times [-1,+1] \arrow[d, dashed] \\
L_\lambda \arrow[r, dashed] & W
\end{tikzcd}$$

and note that

$$\partial W = \partial_0 W \coprod \partial_1 W,$$

where

$$\begin{tikzcd}
\partial_{\mathrm{left}} L_\lambda \cap \mathbb{R}_\times \arrow[r] \arrow[d] & (M \setminus \varphi(\mathbb{S}^{\lambda-1}) \arrow[d, dashed] \\
\partial_{\mathrm{left}} L_\lambda \arrow[r, dashed] & \partial_0 W
\end{tikzcd}$$

and

$$\begin{tikzcd}
\partial_{\mathrm{right}} L_\lambda \cap \mathbb{R}_\times \arrow[r] \arrow[d] & (M \setminus \varphi(\mathbb{S}^{\lambda-1}) \arrow[d, dashed] \\
\partial_{\mathrm{right}} L_\lambda \arrow[r, dashed] & \partial_1 W
\end{tikzcd}$$

so $\partial_0 W \simeq M$ and $\partial_1 W \simeq \mathsf{Surg}(M, \varphi)$.

Hence W is a cobordism between M and $\mathsf{Surg}(M, \varphi)$; to finish we must indicate the pertinent elementary Morse function $f \in \mathsf{Morse}(W)$. But observe that under the above identifications, the smooth map

$$\widetilde{f} : (M \setminus \varphi(\mathbb{S}^{\lambda-1}) \times [-1, +1] \coprod L_\lambda \longrightarrow \mathbb{R}$$
$$\widetilde{f}|_{(M \setminus \varphi(\mathbb{S}^{\lambda-1}) \times [-1,+1]} = \mathrm{pr}_2, \quad \widetilde{f}|_{L_\lambda} = Q_\lambda$$

descends to a smooth $f \in C^\infty(W)$ with the required properties.

□

On the other hand, suppose $(\mathcal{C}, f)$ is an elementary cobordism, where $f : W \to \mathbb{D}^1$ is an elementary Morse function with a unique critical point p of index λ at the level set 0.

We wish to define an embedding

$$\varphi : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \hookrightarrow \partial_0 W$$

Fix a Morse chart $e : B^{m+1}_{2\varepsilon} \hookrightarrow W^{m+1}$ centred at p,

$$e(0) = p \in \mathsf{Crit}(f), \quad e^* f = Q_\lambda.$$

Then

$$\varphi' : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \hookrightarrow f^{-1}(-\varepsilon),$$
$$(u, \theta v) \mapsto e(\sqrt{\varepsilon} u \cosh\theta, \sqrt{\varepsilon} u \sinh\theta)$$

embeds $\mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1}$ in the *regular* level set $f = -\varepsilon$. The (local) flow ϕ^t of the vector field $w := -\frac{\nabla f}{\|\nabla f\|^2} \in \mathfrak{X}(M \setminus p)$, $\phi : (M \setminus p) \times \mathbb{R} \supset \mathrm{dom}(\phi) \longrightarrow M \setminus p$, determines a homotopy of embeddings

$$\varphi'_t : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \hookrightarrow W,$$
$$\mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \times [0, 1 - \sqrt{\varepsilon}] \to W, \quad ((u, \theta v), t) \mapsto \phi^{-t}(\varphi'(u, \theta v)),$$

and we set

$$\varphi := \varphi'_{1-\sqrt{\varepsilon}} : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \hookrightarrow \partial_0 W$$

Observe that the choice of $\varepsilon > 0$ is immaterial, since the embeddings determined by any two choices according to the recipe above must coincide.

By the same token, we can drag the embedding

$$\Phi' : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \hookrightarrow f^{-1}(\varepsilon), \quad (u, \theta v) \mapsto e(\sqrt{\varepsilon} u \sinh\theta, \sqrt{\varepsilon} u \cosh\theta)$$

along the flow of w from time $t = 0$ to $t = 1 - \sqrt{\varepsilon}$ to obtain an embedding

$$\Phi : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \hookrightarrow \partial_1 W.$$

Definition 17. *We call the embeddings φ, Φ* **characteristic-** *and* **cocharacteristic embeddings** *of $(\mathcal{C}, f)$.*

Remark 3. *Note that the (co-)characteristic embedding depends on the choice of Morse chart, and also on the vector field ∇f which we used to drag objects around. Such choices will be implicit whenever we speak of such embeddings.*

Theorem 12. *If $(\mathcal{C}, f)$ is elementary of index λ, then $\partial_1 W \simeq \mathsf{Surg}(\partial_0 W, \varphi)$, for some characteristic embedding $\varphi : \mathbb{S}^{\lambda-1} \times \mathring{\mathbb{D}}^{m-\lambda+1} \hookrightarrow \partial_0 W$.*

Proof. In terms of the notation above, one argues as in Theorem 11 to deduce that $f^{-1}(\varepsilon) \simeq \mathsf{Surg}(f^{-1}(-\varepsilon), \varphi')$, and $\partial_0 W \simeq f^{-1}(-\varepsilon)$, $\partial_1 W \simeq f^{-1}(\varepsilon)$ under $\phi^{\pm(\sqrt{e}-1)}$. □

Let $(\mathcal{C}, f)$ be an elementary cobordism of index λ, with characteristic and characteristic embeddings φ, Φ, respectively.

Definition 18. *The* **core disk** $\mathsf{Core}_\lambda(p)$ *of the critical point p is the union of trajectories of ∇f beginning in $\varphi(\mathbb{S}^{\lambda-1}) \subset \partial_0 W$ and ending at p.*

Its **cocore disk** $\mathsf{Cocore}^{m-\lambda}(p)$ *is the union of trajectories of ∇f beginning in p and ending in $\Phi(\mathbb{S}^{m-\lambda}) \subset \partial_1 W$.*

Note that it follows from the above discussion that these are *smoothly* embedded disks, meeting transversally at p, and determining the decomposition

$$T_p W = T_p \mathsf{Core}_\lambda(p) \oplus T_p \mathsf{Cocore}^{m-\lambda}(p)$$

into negative-definite and positive-definite subpaces for $\mathsf{Hess}_p(f)$.

Corollary 5. *If $(\mathcal{C}, f)$ be an elementary cobordism of index λ,*

$$(\partial_0 W \cup \mathsf{Core}_\lambda(p)) \hookrightarrow \partial_1 W$$

is a deformation retraction. In particular

$$H_\bullet(W, \partial_0 W; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = \lambda; \\ 0 & \text{otherwise.} \end{cases}.$$

and so the index of an elementary cobordism $\mathcal{C}$ is independent of the choice of elementary Morse function.

4.2.1. *Exercises.*

(1) A **gradient-like** vector field for $f \in \mathsf{Morse}(M)$ is a $w \in \mathfrak{X}(M)$ such that :
- $wf > 0$ on $M \setminus \mathsf{Crit}(f)$;
- For every $p \in \mathsf{Crit}(f)$, there is a Morse chart $e : B_{2\varepsilon} \hookrightarrow M$ centred at p, pulling w back to

$$e^* w = -2 \sum_{1}^{\lambda} x_i \frac{\partial}{\partial x_i} + 2 \sum_{\lambda+1}^{n} y_i \frac{\partial}{\partial y_i}.$$

(a) Convince yourself that, except for Lemma 4, all arguments involving the gradient ∇f with respect to some Riemannian metric remain true if ∇f is replaced by a gradient-like vector field w.

(b) Let w be a gradient-like vector field for f Morse on the *compact* manifold M, and let $\varphi : M \times \mathbb{R} \to M$ denote its flow. For any $x \in M$, let $\omega(x)$ be the collection of those points of M which are limit points sequences of the form $(\phi^{t_n}(x))_{n \geqslant 0}$, where $t_n \to +\infty$. Show that $\omega(x)$ is contained in a level set of f. Similarly, the limit points $\alpha(x)$ to sequences of the form $(\phi^{t_n}(x))_{n \geqslant 0}$, $t_n \to -\infty$, lie in a single level set of f.

(c) Show that $\alpha(x)$ and $\omega(x)$ are invariant under the flow of w.

(d) Show that $\alpha(x) \subset \mathsf{Crit}(f) \supset \omega(x)$.

(e) Show that $\alpha(x) = \{p\}$ and $\omega(x) = \{q\}$. Conclude that, for every $x \in M$, $\lim_{t \to \pm\infty} \phi^t(x)$ exists and is a critical point.

(2) Prove Corollary 5.

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