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Poisson Geometry - Global Aspects

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0. Origins

There were, originally, two main motivations to study Poisson brackets:

- Hamiltonian Dynamics and the search for first integrals
- Quantization of classical mechanical systems

Later many connections with our subjects (e.g., harmonic analysis, representation theory, singularity theory, control theory, etc.) were discovered and Poisson Geometry become a subject on its own.

· **Def**: A Poisson bracket on a manifold M is a Lie bracket $\{\cdot,\cdot\}: C^\infty(M)\times C^\infty(M) \to C^\infty(M)$ satisfying the Leibniz identity

$$\{f, gh\} = \{f,g\}h + g\{f,h\}$$

The pair $(M, \{\cdot,\cdot\})$ is called a Poisson manifold. A map $\phi: M \to N$ between Poisson manifolds $(M, \{\cdot,\cdot\}_M)$ & $(N, \{\cdot,\cdot\}_N)$ is a Poisson map if

$$\{f\circ\phi, g\circ\phi\}_M = \{f,g\}_N\circ\phi \quad , \quad \forall f,g\in C^\infty(N).$$

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Historical Note:

· Siméon Denis-Poisson (1781-1840) was a French mathematician. Check Wikipedia for his many contributions, in particular in Celestial Mechanics to the problem of stability of planetary orbits. Although Poisson brackets were study by Jacobi and S. Lie, the formal definition is due to A. Lichnerowicz (70's).

- Why should we really care about such objects?
- Are there natural examples?
- What kind of questions arise in Poisson Geometry?

Example 1 (Symplectic manifolds and related objects)

- (S,ω) - symplectic manifold
- For $h \in C^\infty(S)$, let $X_h \in \mathfrak{X}(S)$ be associated hamiltonian vector field:

$$i_{X_h}\omega = dh$$

Then one defines a Poisson bracket:

$$\{f,g\} := -\omega(X_f, X_g)$$

Exercise 1. Check that this defines a Poisson bracket on S and that a diffeomorphism $\phi: S \to S$ is symplectic iff it is a Poisson map.

Hint: Verify the formulas:

$$\{f,g\} = X_f(g) \quad \& \quad [X_f, X_g] = X_{\{f,g\}}$$

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Note (Our sign conventions) Be aware that this varies in the literature! For us, the canonical symplectic structure on T^*M is the unique symplectic form:

$$\alpha^*\omega_0 = -d\alpha, \quad \forall\, \alpha \in \Omega^1(M) = \Gamma(T^*M)$$

Check that with these sign conventions, for $T^*\mathbb{R}^n$ with coordinates (q_i, p_i) we have:

- $\omega_0 = dq_i \wedge dp_i$
- $X_h = \dfrac{\partial h}{\partial p_i}\dfrac{\partial}{\partial q_i} - \dfrac{\partial h}{\partial q_i}\dfrac{\partial}{\partial p_i}$ (use Einstein convention)
- $\{f,g\} = \dfrac{\partial f}{\partial p_i}\dfrac{\partial g}{\partial q_i} - \dfrac{\partial f}{\partial q_i}\dfrac{\partial g}{\partial p_i}$

Assume now that $G \times S \to S$ is a proper & free action by symplectomorphisms: by the exercise, this means that for each $g \in G$:

$$\Phi_g : S \to S, \quad s \mapsto g \cdot s, \text{ is a Poisson diffeomorphism}$$

<u>Proposition</u>: The manifold $M \equiv S/_G$ carries a unique Poisson bracket for which $\pi : S \to S/_G$ is a Poisson map.

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<u>Proof</u>: Observe that $C^\infty(M) \cong C^\infty(S)^G$, the space of G-invariant smooth functions. Moreover, the Poisson bracket of G-invariant functions is a G-invariant function.

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<u>Exercise 2</u>: Check that if $\dim G > 0$, then the Poisson bracket on $M = S/G$ does not come from a symplectic form on M.

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<u>Exercise 3</u>: Let $S = \mathbb{C}^2 \setminus \{0\}$ w/ symplectic form $\omega = \frac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\bar{w})$ and consider the action $S^1 \times S \to S$, $\theta \cdot (z, w) = (e^{i\theta} z, e^{-i\theta} w)$. Determine the Poisson structure on $M = S/_{S^1} \simeq \mathbb{R}^3 \setminus \{0\}$.

<u>Hint</u>: Consider the following generating set of S^1-invariant polynomials

$$G_1 = zw + \bar{z}\bar{w}, \quad G_2 = i(zw - \bar{z}\bar{w}), \quad G_3 = |z|^2 - |w|^2, \quad G_4 = |z|^2 + |w|^2$$

which satisfy the relation:

$$G_4^2 = G_1^2 + G_2^2 + G_3^2$$

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For an arbitrary Poisson manifold $(M, \{\cdot,\cdot\})$ the Leibniz identity shows that we have a Lie algebra homomorphism:

$$C^\infty(M) \ni h \longmapsto X_h \in \mathfrak{X}(M), \quad X_h(f) := \{h, f\}$$

The v.f. X_h is called the hamiltonian vector field associated with $h \in C^\infty(M)$ (~~the~~ sign convention is consistent w/ symplectic case). The study of such vector fields is called Hamiltonian Dynamics.

Example 2: (linear Poisson brackets) Let $\mathfrak{g}$ be any finite dimensional Lie algebra. Then $M = \mathfrak{g}^*$ carries a natural Poisson structure:

$$\{f, g\}(\xi) = \langle \xi, [d_\xi f, d_\xi g]_{\mathfrak{g}} \rangle,$$

where we view $d_\xi f \in (\mathfrak{g}^*)^* \simeq \mathfrak{g}$. For any $h \in C^\infty(\mathfrak{g}^*)$ the dynamics of the corresponding v.f. X_h are given by Euler's equations:

$$\dot{\xi}(t) = -\,ad^*(d_{\xi(t)} h) \cdot \xi(t)$$

Exercise 4: Show that in this example $M = T^*G/G$, where T^*G is equipped with the canonical symplectic structure and $G \times T^*G \to T^*G$ is the cotangent lift of the action of G on itself by translations.

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The Poisson structure in the previous example is linear: if $M = V$ is a vector space, then a Poisson bracket is linear if the bracket of any pair of linear functions is a linear function.

Exercise 5: Show that every linear Poisson bracket on a vector space V is of the form $\mathfrak{g}^*$, for some Lie algebra $\mathfrak{g}$. Also, a linear map $\phi: \mathfrak{g}^* \to \mathfrak{h}^*$ is a Poisson map iff $\phi^*: \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra morphism.

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Example 3 (Quadratic Poisson brackets)

Let $M = \mathbb{R}^n$. Any skew-symmetric matrix $A = (a_{ij})$ determines a Poisson structure, which in linear coordinates $(x_1, .., x_n)$ is given by:

$$\{x_i, x_j\} = a_{ij}\, x_i x_j \quad (\text{no sum!!})$$

If we restrict to the open set $\mathbb{R}^n_+ = \{x_i > 0,\ i = 1..n\}$ and we set

$$h(x_1, .., x_n) = \sum_{j=1}^{n} \left(q_j \log x_j - x_j \right)$$

The the dynamics of the vector field X_h are given by the Lotka-Volterra equations:

$$\dot{x}_i = \varepsilon_i x_i + \sum_{j=1}^{n} a_{ij}\, x_i x_j \qquad (i = 1..n)$$

where $\varepsilon_i = -\sum_{j=1}^{n} a_{ij} q_j$ (note that $q = (q_1, .., q_n)$ is an equilibrium of the system).

Exercise 6: Consider the map $\phi : \mathbb{R}^{2n} \to \mathbb{R}^n_+$ given by

$$(q_1 .., q_n, p_1, .., p_n) \longmapsto x_i = \exp\Big(p_i - \tfrac{1}{2} \sum_{j=1}^{n} a_{ij} q_j \Big)$$

Check that this gives a Poisson map if we take $\omega_0 \in \Omega^2(\mathbb{R}^n)$ the canonical symplectic form. Is this a symmetry reduction for some group action on $\mathbb{R}^{2n}$ by symplectomorphisms?

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In order to integrate (= find orbits) a vector field X it helps to know first integrals, i.e., functions $I : M \to \mathbb{R}$ such that

$$I(x(t)) = \text{constant}$$

$$\Longleftrightarrow \quad \dot{x}^i(t) \cdot \frac{\partial I}{\partial x^i} = 0 \quad \Longleftrightarrow \quad X(I) = 0$$

If we have several independent first integrals we can restrict to the common level sets and, hence, reduce the dimension. For hamiltonian systems we can do better!

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<u>Proposition</u>

Let X_h be a hamiltonian v.f. on a Poisson manfld $(M, \{\cdot,\cdot\})$. Then:

(i) I is a 1st integral of $X_h \iff \{h, I\} = 0$;

(ii) h is always a 1st integral of X_h;

(iii) If I_1 & I_2 are 1st integrals of X_h then $\{I_1, I_2\}$ is also a 1st int.;

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Let I be a 1st integral of X_h. Assume that X_I is complete and that its flow $\varphi^t_{X_I}$ defines an ~~proper & free~~ action of $\mathbb{R}$:

$$\begin{aligned} \mathbb{R} \times M &\longrightarrow M \\ (t, x) &\longmapsto \varphi^t_{X_I}(x) \end{aligned}$$

Note that this action preserves the level sets $\{I(x) = c^{te}\}$.

<u>Proposition</u> ("Hamiltonian Reduction")

Assume the action is proper & free on a neighborhood of level set $I^{-1}(a)$. Then the reduced manifold

$$M_a := I^{-1}(a)/_{\mathbb{R}}$$

carries a natural Poisson structure ~~[illegible]~~ s.t. we have Poisson maps:

$$\begin{tikzcd} M \arrow[dr, "\phi"] & X_h \arrow[dr, mapsto] & & \\ & M/_{\mathbb{R}} & X_{\bar h} & \\ I^{-1}(a)/_{\mathbb{R}} \arrow[ur] & & X_{\bar h}|_{I^{-1}(a)/\mathbb{R}} \arrow[ul, mapsto] & \end{tikzcd} \qquad h = \bar h \circ \phi$$

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<u>Note</u>: The action of $\mathbb{R}$ on $I^{-1}(a)$ ~~being~~ is assumed proper & free ~~implies that we can choose~~ only on an open neighborhood of $I^{-1}(a)$ ~~where action is proper & free~~. We assume, for simplicity that M equals this neighborhood.

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Proof:

By the Jacobi identity:

$$X_I(\{f,g\}) = \{I,\{f,g\}\}$$
$$= \{\{I,f\},g\} + \{f,\{I,g\}\}$$
$$= \{X_I(f),g\} + \{f, X_I(g)\} \quad \left(\text{i.e., } X_I \text{ is a Poisson v.f.}\right)$$

So we conclude that:

$$\{f,g\}\circ \varphi^t_{X_I} = \{f\circ\varphi^t_{X_I}, g\circ\varphi^t_{X_I}\} \quad \left(\text{i.e., } \varphi^t_{X_I}: M\to M \text{ is a Poisson Diff.}\right)$$

$\Rightarrow$ $\mathbb{R}$-invariant functions are closed under $\{\cdot,\cdot\}$

$\Rightarrow$ $M/_{\mathbb{R}}$ has a unique Poisson st. ~~that~~ $\phi: M \to M/_{\mathbb{R}}$ is Poisson.

The rest is more or less obvious, since $\{I,f\}=0$, $\forall f\in C^\infty(M)^{\mathbb{R}}$. ∎

$$(M, \{\cdot,\cdot\}_M, X_h) \xrightarrow{I = 1^{st}\text{ integral}} (N_a, \{\cdot,\cdot\}_{M_a}, \bar{h}) \quad \text{w/ } \dim N_a = \dim M - 2$$

Exercise 7

Show that if J is another first integral & $\{I,J\}=0$

Then there exists $\bar{J}: M/_{\mathbb{R}} \to \mathbb{R}$ such that $\bar{J}|_{M_a}$

is a 1^{st} integral of $X_{\bar{h}|_{M_a}}$

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Defn: A hamiltonian v.f. X_h on a Poisson manifold $(M,\{\cdot,\cdot\}_M)$ is called completely integrable if it admits $\{I_1,\dots,I_r,\dots,I_s\}$ 1^{st} integrals s.t.:

(i) $\{I_j, I_k\}=0$, $1\le j\le r$, $1\le k\le s$.

(ii) $r+s = \dim M$.

(iii) $dI_1\wedge\dots\wedge dI_k \neq 0$ & $X_{I_1}\wedge\dots\wedge X_{I_r}\neq 0$ (a.e.).

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Note

- $I_1, \dots, I_r \Rightarrow$ reduce dim by $2r$
- $I_{r+1}, \dots, I_s \Rightarrow$ reduce dim by $s-r$

$\Big\} \Rightarrow$ reduce dim by $2r+s-r = r+s = \dim M$

$\Rightarrow$ system can be integrated by quadratures

Let us consider the "moment map":

$$\phi: M \to \mathbb{R}^s, \quad x \mapsto (I_1(x), \dots, I_s(x))$$

Proposition

If ϕ is a surjective submersion then there exists a unique Poisson bracket on $\mathbb{R}^s$ such that ϕ becomes a Poisson map

Proof: Exercise. ∎

(Hint: show that the Poisson bracket of functions constant on the fibers is constant on the fibers.)

Exercise 8 (symplectic case)

If M is symplectic then the fibers of $\phi: M \to \mathbb{R}^s$ are isotropic submanifolds [when $r = s$, so all integrals commute with each other,] one obtains a Lagrangian fibration.

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If $\phi: M \to \mathbb{R}^s$ is a Poisson map, with M symplectic, which is also a surjective submersion with isotropic fibers, we will see later that locally we have an integrable system: for every $x \in \mathbb{R}^s$ there exists a neighborhood $U \ni x$ and a diffeomorphism $\psi: U \to \mathbb{R}^s$ such that $\psi \circ \phi: M \to \mathbb{R}^s$ has components 1st integrals as in Definition above.

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Example 4 (Motion in a Central Force Field)

A particle with mass m and coordinates $(q_1, q_2, q_3) \in \mathbb{R}^3$ under a force with potential $V = V(q_1^2+q_2^2+q_3^2)$:

- $M = T^*\mathbb{R}^3$ w/ coordinates $(q_1, q_2, q_3, p_1, p_2, p_3)$ & $\omega = \sum_{i=1}^{3} dq_i \wedge dp_i$
- $h = \sum_{i=1}^{3} \frac{p_i^2}{2m} + V(|q|^2)$

The hamiltonian system X_h admits as 1^a integrals:

- The "energy" h;
- The ~~angular~~ components of the angular momentum $\ell = (q_1,q_2,q_3) \times (p_1,p_2,p_3)$ so $\ell_1 = q_2p_3 - q_3p_2$, $\ell_2 = q_3p_1 - q_1p_3$, $\ell_3 = q_1p_2 - q_2p_1$.

One checks easily that:

- $\{h, \ell_i\} = 0$, $i = 1,2,3$
- $\{\ell_1, \ell_2\} = \ell_3$ & $\{\ell_2, \ell_3\} = \ell_1$ & $\{\ell_3, \ell_1\} = \ell_2$

This means that $\{h, \ell_1, \ell_2, \ell_3\}$ does not satisfy the assumptions of the definition. But if we observe that $\{\ell_1^2+\ell_2^2+\ell_3^2, h\} = 0$, then we conclude we can use $\{h, \ell_1^2+\ell_2^2+\ell_3^2, \ell_2, \ell_3\}$ which satisfy the assumptions.

Note that the maps:

$$M \ni x \longmapsto (h(x), \ell_1(x), \ell_2(x), \ell_3(x)) \in \mathbb{R}^4$$
$$\searrow (h(x), \ell_1^2(x)+\ell_2^2(x)+\ell_3^2(x), \ell_2(x), \ell_3(x)) \in \mathbb{R}^4$$

have the same fibers and differ locally by a diffeomorphism.

The Poisson structure on $\mathbb{R}^4$ induced by the first map identifies $\mathbb{R}^4 \simeq \mathbb{R} \times \mathfrak{so}(3)^*$, as Poisson manifolds.

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The previous discussion should make it clear that the following questions are interesting:

- Given a Poisson manifold $(M, \{\cdot,\cdot\})$ is it of the form $M = S/G$ for some symplectic manifold S and symplectic action $G \times S \to S$?

- Given a Poisson manifold $(M, \{\cdot,\cdot\})$ is there a symplectic manifold (S, ω) & a surjective, Poisson submersion $\phi : S \to M$? Can we require further properties on the fibers (e.g., isotropic) ?

These are two examples of global questions of Poisson geometry. To answer these and other questions we need to understand better the geometry of Poisson brackets.

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