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Lecture 2

In the previous lecture we saw some natural Global Problems that appear in the study of Poisson brackets. In order to deal and, eventually, solve this problems we need to understand the geometry underlying Poisson brackets.

1. Local Poisson Geometry: A Review

Recall that Poisson brackets are more efficiently described using the language of multi-vector fields (objects dual to differential forms).

$$\{\cdot,\cdot\}: C^\infty(M)\times C^\infty(M) \to C^\infty(M) \quad \longleftrightarrow \quad \pi \in \mathfrak{X}^2(M) := \Gamma(\Lambda^2 TM)$$

bilinear, skew-symmetric, Leibniz

$$\{f,g\} = \pi(df, dg)$$

$$\text{Jacobi identity} \quad \longleftrightarrow \quad [\pi,\pi] = 0$$

In local coordinates $(x^1,\dots,x^m)$:

$$\pi = \sum_{i<j} \pi^{ij}(x)\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j} \qquad \text{where } \pi^{ij}(x) = \{x^i, x^j\}$$

— / —

Exercise 1: Check that $\pi \in \mathfrak{X}^2(M)$ defines a bundle map $\pi^\#: T^*M \to TM$. The rank of π at $x_0 \in M$ is defined to be $\dim(\operatorname{Im} \pi^\#_{x_0})$. Check that this number equals $\operatorname{rank}(\pi^{ij}(x_0))$ for any choice of local coordinates.

The rank is an even number. — / —

Theorem (Darboux-Weinstein)

Let x_0 be a point in (M,π) and assume that $\operatorname{rank} \pi_{x_0} = 2\ell$. Then there exist coordinates $(q^1,\dots,q^\ell, p^1,\dots,p^\ell, z^1,\dots,z^s)$ centered at x_0 such that:

$$\pi = \underbrace{\sum_{i=1}^{\ell} \frac{\partial}{\partial p^i}\wedge\frac{\partial}{\partial q^i}}_{\text{Symplectic}} + \underbrace{\sum_{j<k} \varphi^{jk}(z)\frac{\partial}{\partial z^j}\wedge\frac{\partial}{\partial z^k}}_{\text{Poisson vanishing at } x_0}, \quad \text{w/ } \varphi^{jk}(0) = 0.$$

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Proof: (sketch)

Use induction on Rank π_{x_0}:

- If Rank $(\pi_{x_0}) = 0$, Result is obvious
- If Rank $(\pi_{x_0}) > 0 \Rightarrow \exists$ local function p^1 with $X_{p^1}|_{x_0} \neq 0$

We can straighten out X_{p^1}: $\exists$ local function q^1 with $X_{p^1}(q^1) = 1 \Leftrightarrow \{p^1, q^1\} = 1$

Since:
$$[X_{p^1}, X_{q^1}] = X_{\{p^1,q^1\}} = 0$$
we can complete (p^1, q^1) to a coordinate system $(p^1, q^1, x^3, \dots, x^m)$ such that
$$X_{p^1}(x^i) = X_{q^1}(x^i) = 0 \Leftrightarrow \{p^1, x^i\} = \{q^1, x^i\} = 0$$
Finally, note that:
$$0 = \{\{x^i, p^1\}, x^j\} + \{x^i, \{x^j, p^1\}\} = \{\{x^i, x^j\}, p^1\} = \frac{\partial\{x^i, x^j\}}{\partial q^1} \underbrace{\{q^1, p^1\}}_{= -1}$$
$$\Rightarrow \frac{\partial\{x^i, x^j\}}{\partial q^1} = 0$$
Similarly, $\frac{\partial\{x^i,x^j\}}{\partial p^1} = 0$, so we see that $\{x^i, x^j\}$ is independent of (q^1, p^1). Now use induction. $\boxtimes$

Darboux-Weinstein says that M is locally a product of symplectic manifold and a Poisson structure vanishing at the point.

Exercise 2

A (non-degenerate) b-Poisson structure is a Poisson structure $\pi \in \mathfrak{X}^2(M^n)$ such that $\Lambda^n \pi \pitchfork$ zero section. Let $Z = (\Lambda^n \pi)^{-1}$(zero section) and show that

(i) π has rank $2n$ in $M \setminus Z$, so if $x_0 \notin Z$ $\exists$ coordinates $(q^1, p^1, \dots, p^n, q^n)$: $\pi = \sum_{i=1}^{n} \frac{\partial}{\partial p^i} \wedge \frac{\partial}{\partial q^i}$

(ii) If $x_0 \in Z$, show that exists local coordinates $(x, y, p^2, q^2, \dots, p^n, q^n)$ such that
$$\pi = x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \sum_{i=2}^{n} \frac{\partial}{\partial p^i} \wedge \frac{\partial}{\partial q^i}$$

—/—

In the decomposition given by Darboux-Weinstein there is a canonical representative for the symplectic factor, as we now show:

Defn $I = [0,1]$

A <u>cotangent path</u> $a : [0,1] \to T^*M$ is any path s.t.:

$$\pi^\#(a(t)) = \frac{d}{dt} p(a(t)) \quad (\text{where } p : T^*M \to M)$$

The initial / final point of a is $p(a(0)) / p(a(1))$.

$$\begin{tikzcd} & T^*M \arrow[r, "\pi^\#"] \arrow[d, "p"] & TM \arrow[dl] \\ I \arrow[ur, "a"] \arrow[r, "p\circ a"'] \arrow[urr, dashed, "(p\circ a)'"] & M & \end{tikzcd}$$

— / —

<u>Example</u>: If $\gamma : I \to M$ is an integral curve of X_h then $a(t) := d_{\gamma(t)} h$ is a cotangent path.

— / —

<u>Proposition</u>:

Consider the equivalence relation $x \sim y$ if there exists a cotangent path joining x to y. Then the equivalence classes ~~...~~ are (immersed) submanifolds, with a natural symplectic form, such that the inclusion becomes a Poisson map (i.e., they are Poisson submanifolds).

— / —

<u>Proof</u>. Exercise 3. (Hint: use Darboux-Weinstein). ☐

<u>Rmk</u>: Let $i : S \hookrightarrow M$ be such a symplectic leaf. Then:

(i) $d_x i(T_x S) = \operatorname{Im} \pi^\#_x$, i.e., S is an integral submanifold of the (generalized) distribution $x \mapsto \operatorname{Im} \pi^\#_x$.

(ii) S has "better" properties than generic immersed manifolds: For any smooth map $\psi : N \to M$ such that $\psi(N) \subset S$, the induced map $\bar{\psi} : N \to S$ is smooth:

$$\begin{tikzcd} N \arrow[r, "\psi"] \arrow[dr, dashed, "\bar{\psi}"'] & M \\ & S \arrow[u, hook, "i"'] \end{tikzcd}$$

— / —

The symplectic factor has a canonical representative. What about the other factor?

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Exercise 4

Let N be a submanifold of a Poisson manifold (M,π) such that:

(i) $\pi_x^\#(T_xN)^0 \cap T_xN = \{0\}$

(ii) $(T_xN)^0 \cap \operatorname{Ker} \pi_x^\# = \{0\}$

Show that N has a natural induced Poisson structure.

Hint:
(i) + (ii) $\Rightarrow$ $\pi_x^\#(T_xN)^0 \oplus T_xN = T_xM$. Denote by $p: T_xM \to T_xN$ the projection.

Then:
$$T_x^*N \xrightarrow{p^*} T_x^*M \xrightarrow{\pi_x^\#} T_xM \xrightarrow{p} T_xN$$
defines a Poisson structure on N.

— / —

Rmk: - Condition (i) says that N intersects each symplectic leaf transversely. Hence, N is the union of manifolds each of which is a submanifold of a symplectic leaf. Condition (ii) guarantees that these are symplectic submanifolds.

- In general, if $\phi: N \to (M,\pi)$ is a map into a Poisson manifold we can pullback the symplectic foliation:

$$\{\phi^{-1}(S) : S \text{ sympl. leaf of } (M,\pi)\}$$

these will be submanifolds provided $\phi \pitchfork \mathcal{F}$. Each such submanifold has a closed 2-form: the pull back of ω_S. These will be degenerate, in general, and they will form what is called a Dirac Structure.

— / —

Let us fix a symplectic leaf $i: S \hookrightarrow M$. For $x_0 \in S$ let N be a submanifold of M such that $N \pitchfork_{x_0} S$.

Proposition

There exists a neighborhood $x_0 \in U \subset N$ such that (i) & (ii) above hold for all $x \in U$. Hence U has a natural induced Poisson structure.

Proof: Conditions (i) & (ii) are open conditions. $\boxtimes$

Hence, any small enough transverse manifold N to a symplectic leaf S has a Poisson structure.

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Thm

Let $i: S \hookrightarrow (M,\pi)$ be a symplectic leaf and N_1 & N_2 two transversals to S at x_1 & x_2. There exists open neighborhoods $x_1 \in U_1 \subset N_1$ & $x_2 \in U_2 \subset N_2$ and a Poisson diffeo. $\phi: U_1 \to U_2$, with $\phi(x_1) = x_2$.

— / —

Proof (Sketch)

Step 1: Use Darboux-Weinstein to show that the result holds if $x_1 = x_2$.

Step 2: If $x_1 \neq x_2$, choose a cotangent path $a: I \to T^*M$, whose base path $\gamma: I \to M$ is simple & $\gamma(0) = x_1$, $\gamma(1) = x_2$. Then there exists a closed 1-form $\alpha \in \Omega^1(M)$ such that:

$$\alpha|_{\gamma(t)} = a(t)$$

Show that the flow of $X := \pi^\#(\alpha)$ at time 1, gives a local Poisson diffeomorphism that maps a small transversal through x_1 to some small transversal through x_2. Now apply step 1. $\boxtimes$

Examples

1. Let π be a b-Poisson structure (exercise 2). In local coordinates, for a point $x_0 \in Z$, the leaves have codimension 2 and are characterize by $x = c^{te}$ & $y = c^{te}$. The transverse Poisson structure is $\pi^\perp = x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ (which is linear!).

2. For a linear Poisson structure $M = \mathfrak{g}^*$ the symplectic leaves are the coadjoint orbits (why?). In general, it is hard to describe explicitely the transverse Poisson structure (there are several results in the literature about these $\perp$ Poisson structures). As an exercise, you may try the case $\mathfrak{su}(3)^*$.

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We have just seen that cotangent paths play a key role in Poisson geometry. This can be pursued much further: it is one reflection of the fact that in Poisson Geometry T^*M plays the role of TM in ordinary geometry.

Ordinary Geometry

- TM, $\Gamma(TM) = \mathfrak{X}(M)$ has a natural Lie bracket:

$$[X,Y](f) = X(Y(f)) - Y(X(f))$$

with:

$$[X, fY] = f[X,Y] + X(f)Y$$

- Differential forms: $\Omega^\bullet(M) = \Gamma(\Lambda^\bullet T^*M)$

- DeRham differential: $d: \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$

$$d\omega(X_0,\dots,X_n) = \sum_{i=0}^{n} (-1)^i X_i(\omega(X_0,\dots,\check{X}_i,\dots,X_n)) + \sum_{i<j} (-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\check{X}_i,\dots,\check{X}_j,\dots,X_n)$$

$$\Rightarrow H^\bullet_{dR}(M) = \frac{\operatorname{Ker} d}{\operatorname{Im} d} \quad \text{de Rham cohomology}$$

Poisson Geometry

- T^*M, $\Gamma(T^*M) = \Omega^1(M)$ has a natural Lie bracket:

$$[\alpha,\beta] = \mathcal{L}_{\pi^\#(\alpha)}\beta - \mathcal{L}_{\pi^\#(\beta)}\alpha + d(\pi(\alpha,\beta))$$

with:

$$[\alpha, f\beta] = f[\alpha,\beta] + \pi^\#(\alpha)(f)\,\beta$$

- Multivector fields: $\mathfrak{X}^\bullet(M) = \Gamma(\Lambda^\bullet TM)$

- Poisson differential: $d_\pi: \mathfrak{X}^\bullet(M) \to \mathfrak{X}^{\bullet+1}(M)$

$$d_\pi Q(\alpha_0,\dots,\alpha_n) = \sum_{i=0}^{n} (-1)^i \pi^\#(\alpha_i)(Q(\alpha_0,\dots,\check{\alpha}_i,\dots,\alpha_n)) + \sum_{i<j}(-1)^{i+j} Q([\alpha_i,\alpha_j],\alpha_0,\dots,\check{\alpha}_i,\dots,\check{\alpha}_j,\dots,\alpha_n)$$

$$H^\bullet_\pi(M) = \frac{\operatorname{Ker} d_\pi}{\operatorname{Im} d_\pi} \quad \text{Poisson cohomology}$$

Exercise 5

A Poisson vector field is a vector field $X \in \mathfrak{X}(M)$ such that:

$$\mathcal{L}_X \pi = 0 \iff \varphi^t_X \text{ is a Poisson diff for all } t \in \mathbb{R}$$

$$\iff X(\{f,g\}) = \{X(f),g\} + \{f,X(g)\}$$

Show that:

$$H^1_\pi(M) = \frac{\{\text{Poisson v.f.}\}}{\{\text{Hamilt. v.f.}\}}$$

<u>Exercise 6</u>

For any Poisson manifold (M,π) we have that π defines a Poisson cohomology class $[\pi]\in H^2_\pi(M)$. The Poisson structure is called <u>exact</u> if $[\pi]=0$. Show that linear Poisson structures are always exact.

— / —

The computation of Poisson cohomology groups is usually very hard, since there are no general techniques to implement it. Moreover, these groups tend to be infinite dimensional vector spaces. Still, they are quite useful since several canonical classes have a geometric meaning. One such example is the <u>modular class</u> of a Poisson manifold. To define it we use the following exercise:

<u>Exercise 7</u>

Let (M,π) be an orientable Poisson manifold and let $\mu\in\Omega^{top}(M)$ be a volume form. Show that the map $f\mapsto X_\mu(f)$ defined by:

$$\mathcal{L}_{X_f}\mu = X_\mu(f)\,\mu$$

is a derivation, hence we have a well-defined vector field $X_\mu\in\mathfrak{X}(M)$ called the <u>modular vector field</u> associated with μ.

— / —

<u>Proposition</u> (properties of the modular vector field)

(i) X_μ is a Poisson v.f.: $\mathcal{L}_{X_\mu}\pi=0$

(ii) If $\tilde\mu=g\mu$ is another volume form then

$$X_{\tilde\mu} = X_\mu - X_{\log|g|}$$

— / —

<u>Proof</u>:

(i) $X_\mu(\{f,g\})\mu = \mathcal{L}_{X_{\{f,g\}}}\mu = \mathcal{L}_{[X_f,X_g]}\mu = \dots = \big(X_f(X_\mu(g))-X_g(X_\mu(f))\big)\mu$

$= \big(\{f,X_\mu(g)\}+\{X_\mu(f),g\}\big)\mu$

(ii) $X_{\bar\mu}(f)\,\bar\mu = \mathcal{L}_{X_f}\bar\mu = \mathcal{L}_{X_f}(g\mu) = g\,\mathcal{L}_{X_f}\mu + X_f(g)\mu$

$$= g\,X_\mu(f)\,\mu - \frac{1}{g}\{g,f\}\,g\mu$$

$$= \left(X_\mu(f) - X_{\log|g|} \right)\bar\mu \qquad \boxtimes$$

This shows the class $[X_\mu] \in H^1_\pi(M)$ is well-defined and independent of the choice of μ. We call it the modular class of (M,π). Note that:

$$\mathrm{mod}(M) = 0 \iff \exists \text{ volume form } \mu \text{ such that } \mathcal{L}_{X_f}\mu = 0 \text{ for all } f \in C^\infty(M)$$

i.e., $\mathrm{mod}(M)$ is the obstruction for the existence of a volume form invariant under all hamiltonian flows.

Exercise 8

(i) If S is symplectic, $\mathrm{mod}(S) = 0$.

(ii) For a b-Poisson manifold, $\mathrm{mod}(M) \neq 0$.

(iii) What can you say about $\mathrm{mod}(\mathfrak{g}^*)$?

— / —

Exercise 9

Let $G \times M \to M$ be a free action of a (connected) compact Lie group by Poisson diffeomorphisms on (M,π). Show that $\mathrm{mod}(M)=0 \Rightarrow \mathrm{mod}(M/G) = 0$. What if G is not compact? — / —

Examples

In exercise 3, in the previous lecture, we obtain $M = \mathbb{C}^2\setminus\{0\}/S^1 \simeq \mathbb{R}^3 - \{0\}$ with Poisson structure:

$$\pi = (x^2+y^2+z^2)^{1/2}\,\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$$

This Poisson structure has vanishing modular class. On the other hand, the Poisson structure:

$$\tilde\pi = [(x^2+y^2+z^2)^{1/2} - 1]\,\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$$

has non-vanishing modular class. Hence it cannot arise as a quotient of a symplectic manifold by a compact Lie group.

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Theory of Connections

Ordinary Geometry

• **Covariant connection** on v.b. $E \to M$:

$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ $\mathbb{R}$-bilinear s.t.:

(i) $\nabla_X (fs) = f \nabla_X s + X(f) s$

(ii) $\nabla_{fX} s = f \nabla_X s$

• **Covariant derivative along paths**:

Given path $\gamma : I \to M$ and path $u : I \to E$ above γ, then $\nabla_{\dot\gamma} u : I \to E$ is:

$$(\nabla_{\dot\gamma} u)(t) := (\nabla_{\dot\gamma} s_t)(\gamma(t)) + \left(\frac{d s_t}{dt}\right)(\gamma(t))$$

for any time dependent section $s_t \in \Gamma(E)$ such that $u(t) = s_t(\gamma(t))$.

• **//-transport along paths**:

For any path $\gamma : I \to M$,

$$\tau_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$$
$$e \longmapsto u_e(1)$$

where $u_e : I \to E$ is the solution of

$$\begin{cases} u_e(0) = e \\ \nabla_{\dot\gamma} u_e = 0 \end{cases}$$

• **Curvature**: $\Omega_\nabla \in \Omega^2(M, \mathrm{End}(E))$

$$\Omega_\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

• For $E = TM$, **Torsion** $T_\nabla \in \Omega^2(M, TM)$

$$T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

Poisson Geometry

• **Contravariant connection** on v.b. $E \to (M, \pi)$

$\nabla : \Omega^1(M) \times \Gamma(E) \to \Gamma(E)$ $\mathbb{R}$-bilinear s.t.:

(i) $\nabla_\alpha (fs) = f \nabla_\alpha s + \pi^\#(\alpha)(f) s$

(ii) $\nabla_{f\alpha} s = f \nabla_\alpha s$

• **Contravariant derivative along cotangent paths**:

Given cotangent path $a : I \to T^*M$ with base path γ and path $u : I \to E$ above γ, then $\nabla_a u : I \to E$ is:

$$(\nabla_a u)(t) := (\nabla_a s_t)(\gamma(t)) + \left(\frac{d s_t}{dt}\right)(\gamma(t))$$

for any (...)

• **//-transport along cotangent paths**

For any cotangent path $a : I \to T^*M$ w/ base path γ:

$$\tau_a : E_{\gamma(0)} \to E_{\gamma(1)}$$
$$e \longmapsto u_e(1)$$

where $u_e : I \to E$ is the solution of

$$\begin{cases} u_e(0) = e \\ \nabla_a u_e = 0 \end{cases}$$

• **Curvature**: $\Omega_\nabla \in \mathfrak{X}^2(M, \mathrm{End}\, E)$

$$\Omega_\nabla(\alpha, \beta) = [\nabla_\alpha, \nabla_\beta] - \nabla_{[\alpha,\beta]}$$

• For $E = T^*M$, **Torsion** $T_\nabla \in \mathfrak{X}^2(M, T^*M)$

$$T_\nabla(\alpha, \beta) = \nabla_\alpha \beta - \nabla_\beta \alpha - [\alpha, \beta]$$

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Exercise 10

Let ∇ be a covariant connection on T^*M.

(i) Show that the following formulas define contravariant connections on TM & T^*M:

$$\overline{\nabla}_\alpha X := \pi^\#(\nabla_X \alpha) + [\pi^\#(\alpha), X] \qquad \overline{\nabla}_\alpha \beta := \nabla_{\pi^\#\beta}\alpha + [\alpha, \beta]$$

(ii) Show that these connections are intertwined by $\pi^\#$:

$$\overline{\nabla}_\alpha(\pi^\#(\beta)) = \pi^\#(\overline{\nabla}_\alpha \beta)$$

(iii) Aditionally, if $T_\nabla = 0$, show that if $a: I \to T^*M$ is a cotangent path above $\gamma: I \to M$ and $\theta: I \to T^*M$ & $v: I \to TM$ are any paths above γ, then:

$$\langle \overline{\nabla}_a \theta, v \rangle + \langle \theta, \overline{\nabla}_a v \rangle = \frac{d}{dt}\langle \theta, v \rangle$$

—/—

Exercise 11

Given a metric g on T^*M show there exists a unique contravariant connection ∇ on T^*M such that:

(i) ∇ preserves g: $\pi^\#(\alpha) \cdot g(\beta, \gamma) = g(\nabla_\alpha \beta, \gamma) + g(\beta, \nabla_\alpha \gamma)$

(ii) $T_\nabla = 0$

This connection is called the contravariant Levi-Civita metric

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Given a contravariant connection on T^*M a <u>geodesic</u> is a cotangent path $a: I \to T^*M$ such that $\nabla_a a = 0$. Just like in the classical case, geodesics are the integral curves of a vector field $V_\nabla \in \mathfrak{X}(T^*M)$, called the "geodesic spray". This can be used to prove existence of sympl. realizations!

<u>Theorem</u> (Crainic & Marcut, 2010)

Given a contravariant connection ∇ on T^*M w/ geodesic spray V_∇, there exists an open neighborhood $\mathcal{U} \subset T^*M$ of the zero section such that:

$$\omega := \int_0^1 (\varphi^t_{V_\nabla})^* \omega_{can}\, dt$$

is symplectic in $\mathcal{U}$ and the projection $p: T^*M \to M$ is a Poisson map.

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<u>Note:</u> Existence of symplectic realizations go back to 80's by Weinstein et al.