Poisson geometry of moduli spaces of meromorphic connections on curves and their Stokes data

P. Boalch (ENS Paris)

Complex character varieties

(G = connected complex reductive gp)

$$\Sigma \longmapsto \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

Riemann surface — Poisson variety

Atiyah-Bott, Goldman, Karshon, Fock Rosly, Weinstein,
Guruprasad-Huebschmann-Jeffrey-Weinstein, Andersen-Mattes-Reshetikhin ...

generic symplectic leaves are hyperkähler manifolds (Hitchin)

Quasi-Hamiltonian approach

Say $\partial\Sigma = \partial_1 \sqcup \ldots \sqcup \partial_m$ $\quad (\partial_i \cong S^1)$

Choose basepoints $b_i \in \partial_i$

Let $\Pi = \pi_1(\Sigma, \{b_1, \ldots, b_m\})$

Thm (Alekseev et al) $\quad \mathrm{Hom}(\Pi, G)$ is a smooth affine variety

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& symplectic leaves are $\mu^{-1}(e)/G^m$ $\quad (e = (e_1, \ldots, e_m) \subset G^m)$

Further if $\Sigma \longrightarrow \mathbb{B}$ is a family of Riemann surfaces $\Sigma_p \quad p \in \mathbb{B}$

get algebraic Poisson action

$$\pi_1(\mathbb{B}, p) \circlearrowleft \mathrm{Hom}(\pi_1(\Sigma_p), G)/G$$

"The Betti moduli spaces $M_B(\Sigma_p, G) = \mathrm{Hom}(\pi_1(\Sigma_p), G)/G$ form a local system of varieties" (Simpson)

$\Sigma = \overline{\Sigma} \setminus \{a_1, \ldots, a_m\}$ punctured smooth algebraic curve$/\mathbb{C}$

$G = GL_n(\mathbb{C})$

$\mathrm{Hom}(\pi_1(\Sigma), G) / G$

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C^∞ Riemann-Hilbert

$$\left\{ (V, \nabla) \;\middle|\; \begin{array}{l} V \to \Sigma \ \ C^\infty \text{ vector bundle} \\ \nabla \ \ \text{flat connection} \end{array} \right\} \Big/ \text{isom.} \xleftarrow[\;\simeq\;]{} \!\!\!\!\xrightarrow{} \operatorname{Hom}(\pi_1(\Sigma), G) / G$$

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Deligne–Plemelj

Riemann–Hilbert

$$\left\{ (V, \nabla) \;\middle|\; \begin{array}{l} V \to \Sigma \text{ alg. vector bundle} \\ \nabla \text{ alg. connection} \\ \text{with regular sing.s} \end{array} \right\} \Big/ \text{isom.} \quad \xleftrightarrow{\ \cong\ } \quad \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

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$$\Big\downarrow \text{ restrict to } \Sigma$$

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- can now study transcendental aspects of RH map

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- Irregular Atiyah-Bott (– '99, '01)
- Hyperkähler metrics (Biquard-B, '04)
- today: Poisson/sympl. structures algebraically

Definition A *holomorphic quasi-Hamiltonian G-space* is a complex G-manifold M with a G-invariant two form ω and a G-equivariant map $\mu : M \longrightarrow G$ (G acts on G by conjugation) such that

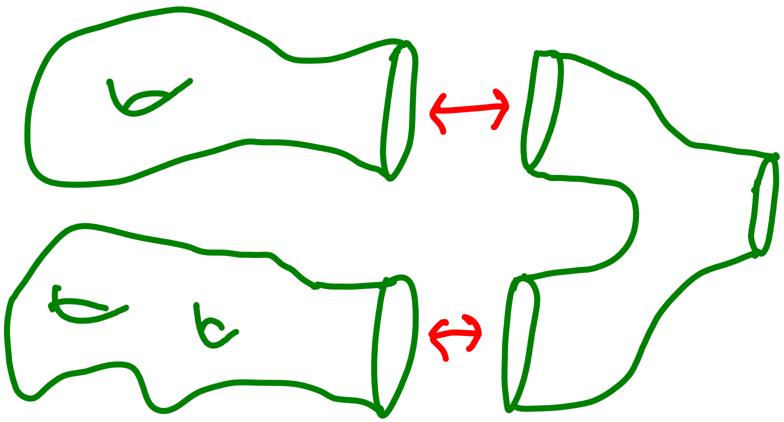
1. $d\omega = \mu^*(\eta)$
2. $\forall X \in \mathfrak{g}$ $\quad \omega(v_X, \cdot) = \frac{1}{2}\mu^*(\theta + \bar{\theta}, X)$
3. $\forall m \in M$ $\quad \ker \omega_m \cap \ker d\mu = \{0\} \subset T_m M$

where η = biinvariant 3-form on G, $\theta, \bar{\theta}$ Maurer-Cartan forms on G

- These axioms are 'what we get from ∞-d viewpoint'
- Multiplicative analogue of Hamiltonian G-space (with $\mathfrak{g}^*$-valued moment map)

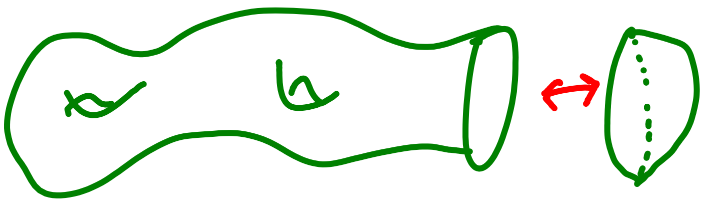
Operations

① Can 'fuse' 2 qHamiltonian G-spaces:



$\Rightarrow$ New qH G-space $M_1 \circledast M_2$

② & reduce:



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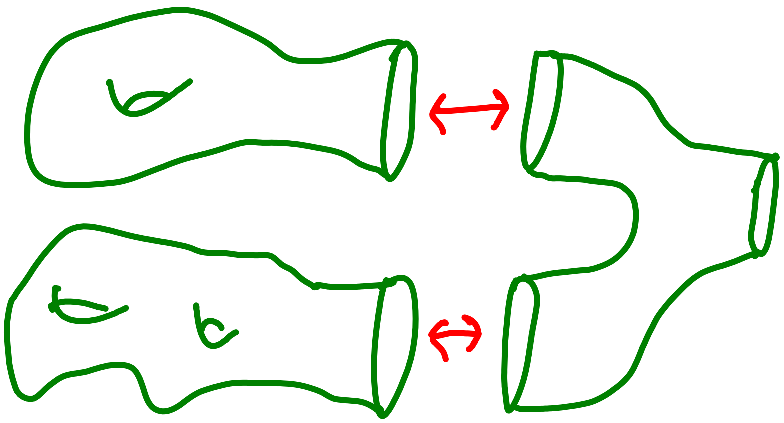

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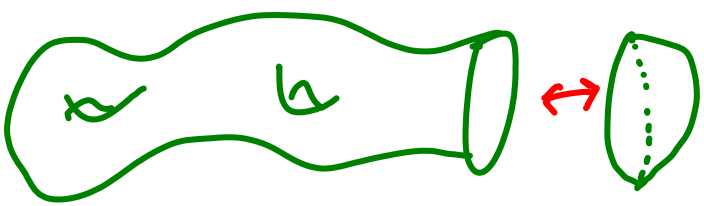

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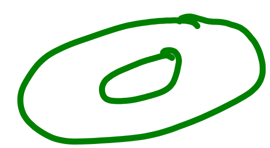
Basic examples

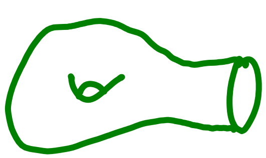
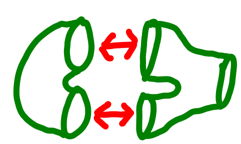
① Conjugacy classes $\mathcal{C} \subset G$

② $D = G \times G$ qH $G \times G$ space (double)



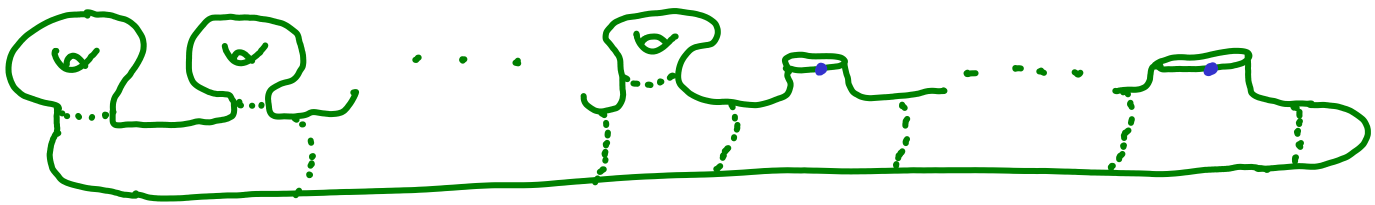
Annulus

③ $\mathbb{D} = G \times G$ qH G-space (internally fused double)

 = 

Can construct all moduli spaces of holomorphic connections on Riemann surfaces from these pieces:

$$\underbrace{\mathbb{D} \circledast \cdots \circledast \mathbb{D}}_{g} \circledast \underbrace{D \circledast \cdots \circledast D}_{m} /\!\!/ G \cong \mathrm{Hom}(\pi, G)$$



$$\mu^{-1}(e)/G^m \cong \left\{ (\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{M}) \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] \prod_{i=1}^{m} M_i = 1 \;,\; M_i \in \mathcal{C}_i \right\} / G$$

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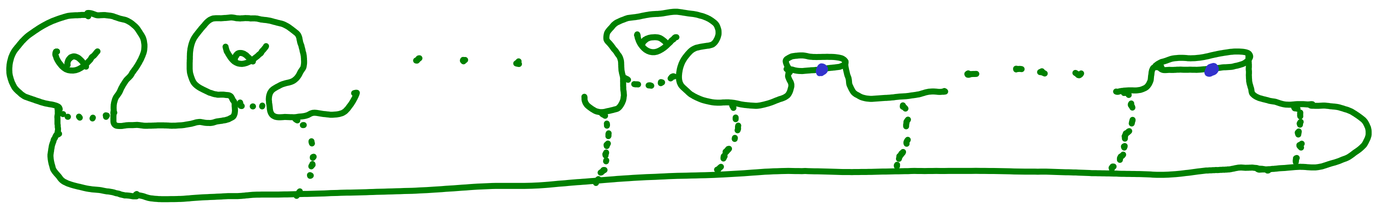

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Aim: New pieces to construct irregular Betti spaces?

New holomorphic q-Hamiltonian spaces

Choose $\quad B_\pm \subset G \quad$ opposite Borel subgroups

$\quad T = B_+ \cap B_- \quad$ maximal torus

$\quad U_\pm \subset B_\pm \quad$ unipotent radicals

New holomorphic q-Hamiltonian spaces

Choose $B_\pm \subset G$ opposite Borel subgroups

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$U_\pm \subset B_\pm$ unipotent radicals

e.g. $G = GL_n(\mathbb{C})$

$$U_+ \sim \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$U_- \sim \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$$

$$T \sim \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

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Thm (– '02)

The space $\,_G\mathcal{A}^r_T := G \times (U_+ \times U_-)^r \times T$

is a quasi-Hamiltonian $G \times T$ space for any $r = 1, 2, \ldots$

- moment map $\mu(C, S_1, \ldots, S_{2r}, t) = (C^{-1} t S_{2r} \cdots S_1 C,\ t^{-1})$
- $\,_G\mathcal{A}^1_T / G \cong G^*$ as Poisson space (dual Poisson Lie group)

non commutative
non cocommutative

$$
\begin{tikzcd}
 & U_q \mathfrak{g} \arrow[dl, "\text{Deconcini-Kac-Procesi}"'] \arrow[dr, "\text{Drinfeld-Jimbo}"] & \\
\mathrm{Fun}(G^*) \arrow[dr] & & U\mathfrak{g} \arrow[dl] \\
 & \mathrm{Fun}(\mathfrak{g}^*) = \mathrm{Sym}(\mathfrak{g}) &
\end{tikzcd}
$$

commutative
noncocommutative

non commutative
cocommutative

commutative
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New holomorphic q-Hamiltonian spaces

Van den Bergh spaces:

Choose V, W complex vector spaces

$$\mathcal{B}(V,W) = \{(a,b) \in \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W) \mid \det(1+ab) \neq 0\}$$

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Thm (M. Van den Bergh '04, '07)

$\mathcal{B}(V,W)$ is a q-Hamiltonian $GL(V) \times GL(W)$ space

- moment map $\mu(a,b) = ((1+ab)^{-1}, 1+ba)$
- defines holom. symplectic structures on multiplicative quiver varieties
- such spaces classify regular holonomic $\mathcal{D}$-modules on a disc

Fission spaces

Choose $P_\pm \subset G$ opposite parabolics

$H = P_+ \cap P_-$ Levi subgroup

$U_\pm \subset P_\pm$ unipotent radicals

Thm (– '02, '09, '11)

The "fission space" $_G\mathcal{A}^r_H := G \times (U_+ \times U_-)^r \times H$

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- moment map $\mu(C, S_1, \ldots, S_{2r}, h) = (C^{-1} h S_{2r} \cdots S_1 C, h^{-1})$
- $(U_+ \times U_-)^r \cong$ Stokes data of meromorphic connections with one level

Example

Suppose $G = GL(V \oplus W)$, $H = GL(V) \times GL(W)$, $U_+ = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \cong Hom(W,V)$

$$U_- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \cong Hom(V,W)$$

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Then ${}_G\mathcal{A}^2_H /\!\!/ G$

$$\cong \left\{ (S_1, S_2, S_3, S_4, h) \in U_+ \times U_- \times U_+ \times U_- \times H \;\middle|\; h S_4 S_3 S_2 S_1 = 1 \right\}$$

is a qHamiltonian $H = GL(V) \times GL(W)$ space $\quad (\mu = h^{-1})$

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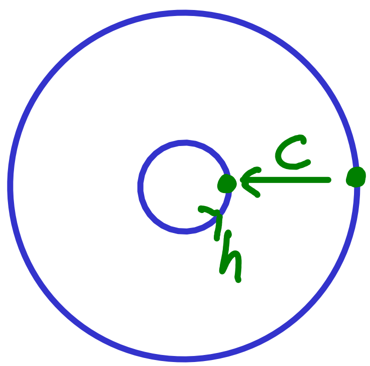
If we write $S_1 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $S_2 = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$, $h = \begin{pmatrix} x & \\ & y \end{pmatrix}$

then

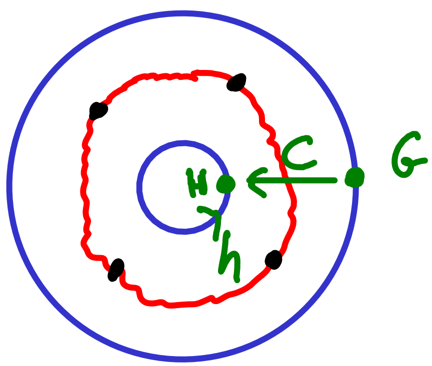
$$x = 1 + ab, \quad y = (1 + ba)^{-1} \quad \& \quad \,_G\mathcal{A}^2_H /\!/ G \cong \mathcal{B}(V,W)$$

Picture If $P_{\pm} = G = H$ $\quad {}_G\mathcal{A}_H = G \times G$ is the double
$$\begin{gathered} G \times G \\ \Downarrow \\ (c, h) \end{gathered}$$



$$\mu = (c^{-1}hc,\ h^{-1})$$

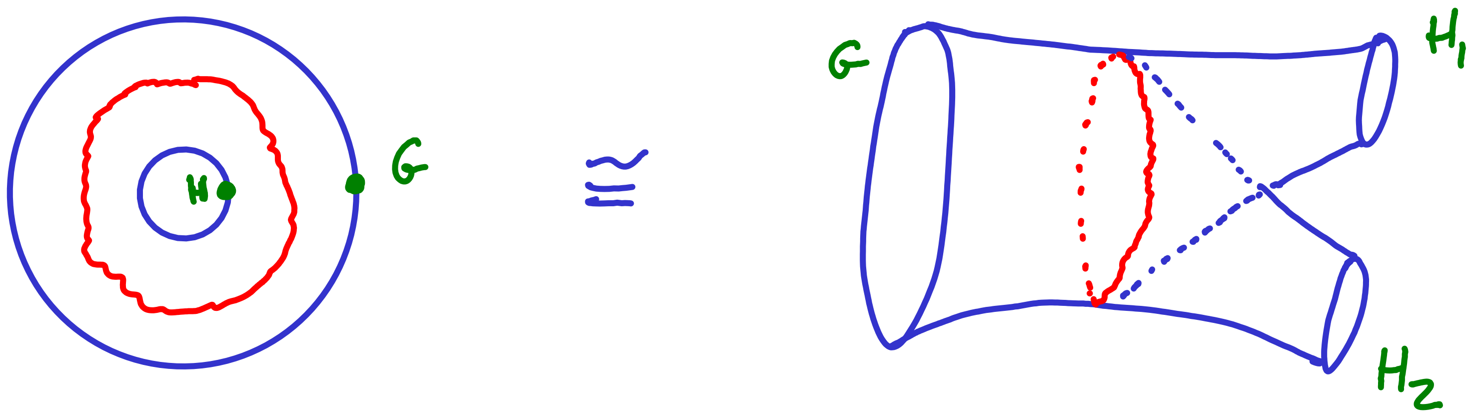
General case can be pictured similarly (breaking group from G to H)



$$\mu = (c^{-1}h\, s_{2r} \cdots s_1\, c,\ h^{-1})$$

Typically H is a product e.g. $H = H_1 \times H_2$

- can glue on both a qH H_1-space & a qH H_2-space



"fission" operation ($\neq$ fusion)

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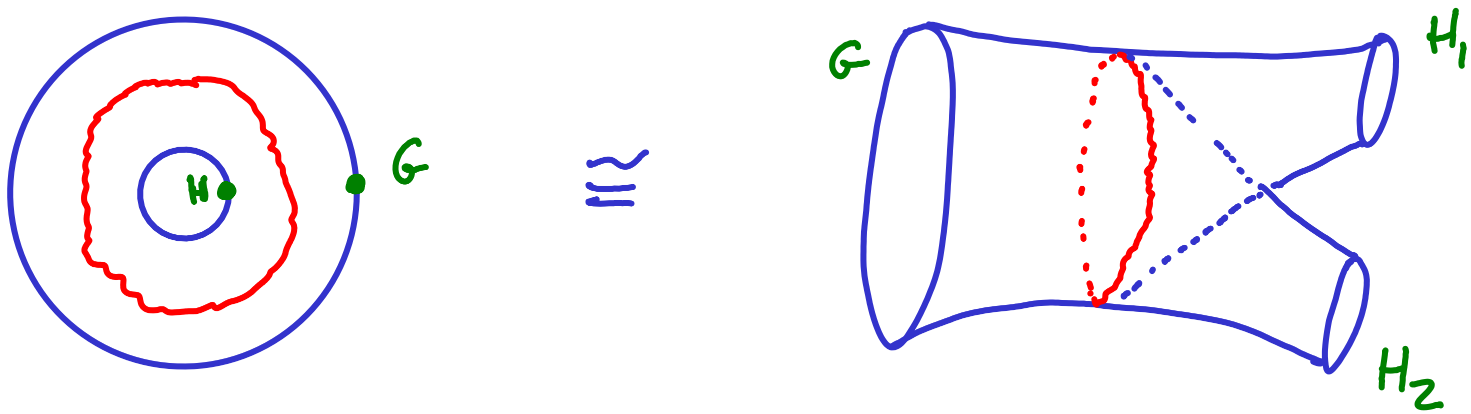

"fission" operation ($\neq$ fusion)

Fusion $\Rightarrow$ induction w.r.t genus & number of marked points

Fission $\Rightarrow$ induction w.r.t order of poles (number of 'levels')

$\mathcal{O} \subset \mathfrak{g}^*$ $\quad$ $G = GL_n(\mathbb{C}) \supset T = (\mathbb{C}^*)^n$

$\mathcal{O} /\!/_1 T$ $\quad$ "complex weight variety"

$\mathcal{O} \subset \mathfrak{g}^*$ $\qquad G = GL_m(\mathbb{C}) \supset T = (\mathbb{C}^*)^m$

$$\mathcal{O} /\!\!/_{\lambda} T \xleftrightarrow{\;\simeq\;} \mathcal{O}_1 \times \dots \times \mathcal{O}_m /\!\!/_{\check{\mathcal{O}}} GL_n(\mathbb{C})$$

Gelfand-MacPherson$^{\mathbb{C}}$, AHH, Hurtubise, Harnad

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Fourier - Laplace

$$\begin{array}{ccc} \mathcal{O} /\!\!/ T & & \mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!\!/ GL_n(\mathbb{C}) \ni \check{\mathcal{O}} \\ \Lambda & & \\ \cup & & \Downarrow \end{array}$$

$$\left(\frac{A_0}{z^2} + \frac{B}{z}\right)dz, \quad B \in \mathcal{O} \longleftrightarrow \sum_1^m \frac{A_i}{z-a_i}\,dz, \quad A_i \in \mathcal{O}_i$$

$A_0 = \mathrm{diag}(a_i)$

$\mathcal{O} \subset \mathfrak{g}^*$ $\qquad$ $G = GL_m(\mathbb{C}) \supset T = (\mathbb{C}^*)^m$

$$\begin{tikzcd}
\mathcal{O}/\!\!/_{1} T \arrow[r, leftrightarrow, "\cong", "\substack{\text{Gelfand-MacPherson}^{\mathbb{C}},\ \text{AHH},\ \text{Harnad} \\ \text{Fourier - Laplace}}"'] & \mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!\!/_{\check{\mathcal{O}}} GL_n(\mathbb{C})
\end{tikzcd}$$

$$\begin{tikzcd}
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$}
\end{tikzcd}$$

$$\begin{tikzcd}
\left(\frac{A_0}{z^2} + \frac{B}{z}\right)dz,\ \substack{B \in \mathcal{O} \\ A_0 = \mathrm{diag}(a_i)} \arrow[r, leftrightarrow] & \sum_{1}^{m} \frac{A_i}{z-a_i}\, dz,\ A_i \in \mathcal{O}_i \arrow[d, "\substack{\text{Riemann} \\ \text{Hilbert}}"'] \\
 & \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m /\!\!/_{\check{\mathcal{C}}} GL_n(\mathbb{C})
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\left(\frac{A_0}{z^2} + \frac{B}{z}\right) dz, \quad B \in \mathcal{O}, \ A_0 = \mathrm{diag}(a_i) \arrow[u, phantom, "\in" rotate=90] \arrow[rr, leftrightarrow] & & \sum_{1}^{m} \frac{A_i}{z - a_i}\, dz, \quad A_i \in \mathcal{O}_i \arrow[u, phantom, "\in" rotate=90] \arrow[d, "\text{Riemann Hilbert}"] \\
\mathcal{C} /\!/_{\lambda} T \arrow[rr, leftrightarrow, "?"] & & \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m /\!/_{\check{\mathcal{C}}} GL_n(\mathbb{C})
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Fourier – Laplace

$$\cup \qquad\qquad\qquad\qquad\qquad\qquad \cup$$

$$\left(\frac{A_0}{z^2} + \frac{B}{z}\right) dz, \quad B \in \mathcal{O}, \quad A_0 = \mathrm{diag}(a_i) \quad \longleftrightarrow \quad \sum_1^m \frac{A_i}{z - a_i}\, dz, \quad A_i \in \mathcal{O}_i$$

Riemann Hilbert $\downarrow$

$$\left(\mathcal{C} \times_{\mathcal{C}} {}_G\mathcal{A}'_T\right) /\!\!/_{\lambda} T \quad \xleftrightarrow{\;?\;} \quad \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m /\!\!/_{\check{\mathcal{C}}} GL_n(\mathbb{C})$$

$\mathcal{O} \subset \mathfrak{g}^*$ $\qquad G = GL_m(\mathbb{C}) \supset T = (\mathbb{C}^*)^m$

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\mathcal{O} /\!\!/_{\Lambda} T \arrow[r, leftrightarrow, "\cong", "\text{Gelfand-MacPherson}^{\mathbb{C}},\ \text{AHH},\ \text{Harnad}"'] \arrow[d, phantom, "\cup" description] & \mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!\!/_{\check{\theta}} GL_n(\mathbb{C}) \arrow[d, phantom, "\cup" description] \\
\left(\dfrac{A_0}{z^2} + \dfrac{B}{z}\right) dz, \quad B \in \mathcal{O},\ A_0 = \mathrm{diag}(a_i) \arrow[r, leftrightarrow, "\text{Fourier – Laplace}"] \arrow[d, "\text{Stokes}"] & \displaystyle\sum_1^m \frac{A_i}{z-a_i}\, dz, \quad A_i \in \mathcal{O}_i \arrow[d, "\text{Riemann Hilbert}"] \\
\Big(\underbrace{\mathcal{C} \circledast {}_G\mathcal{A}'_T}_{\substack{\cong \\ \mathcal{L} \subset G^*}}\Big) /\!\!/_{\lambda} T \arrow[r, leftrightarrow, "\cong"] & \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m /\!\!/_{\check{e}} GL_n(\mathbb{C})
\end{tikzcd}
$$

Société Mathématique de France
2e série, Mémoire n° 15, 1984,p. 129-138

ON THE SIMILARITY BETWEEN THE IWASAWA PROJECTION AND THE DIAGONAL PART

by

J.J. Duistermaat

1. Statement of the result.

Let G be a real connected semisimple Lie group with finite center and $G = KAN$ its Iwasawa decomposition. Via the adjoint representation, and with respect to a suitable basis in $\mathfrak{g}$, K, resp. A, resp. N are the set of matrices in G which are orthogonal, resp. diagonal with positive entries, resp. upper triangular.

The Iwasawa projection H from G onto the Lie algebra $\mathfrak{a}$ of A is defined by

$$(1.1) \qquad x \in K.\exp H(x).N, \quad x \in G.$$

Wild character varieties

(G = connected complex reductive gp)

$$\Sigma \longmapsto \mathrm{Hom}_{\mathbb{S}}(\Pi, G)/\underset{\sim}{H}$$

Irregular curve

Poisson variety

Fix $T \subset G$, Lie algebras $\mathfrak{t} \subset \mathfrak{g}$

Def^{n} Δ complex disc, $a \in \Delta$

An "irregular type" at a is an element

$$Q \in \mathfrak{t}(\hat{K}) / \mathfrak{t}(\hat{\mathcal{O}})$$

[if z local coord vanishing at a, $\hat{K} = \mathbb{C}((z))$, $\hat{\mathcal{O}} = \mathbb{C}[\![z]\!]$

so $$Q = \frac{A_r}{z^r} + \cdots + \frac{A_1}{z} \quad \text{for some } A_i \in \mathfrak{t}$$]

Def^n An "irregular curve" Σ is a smooth compact cx algebraic curve, with distinct marked points $a_1, \dots, a_m \in \Sigma$ and an irregular type Q_i at each marked point

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Aim: define (Poisson) Betti moduli spaces $\mathcal{M}_B(\Sigma, G)$ of irreg. curves Σ & show they form local system of Poisson varieties if Σ undergoes an admissible deformation.

- such spaces will classify mero. connections locally isom. to

$$dQ_i + \text{less singular terms}$$

at each marked point

- local solutions involve $\exp(Q_i)$

Irregular Betti spaces

Irreg RH on curves worked out decades ago for $G = GL_n(\mathbb{C})$

(Birkhoff Balser Jurkat Lutz Malgrange Sibuya Deligne Martinet Ramis ...)

- will give explicit as possible approach using groupoids (for any reductive G)

Irregular Betti spaces

Let Σ be an irreg. curve (marked points $a_1, \ldots, a_m$, irreg. types $Q_1, \ldots, Q_m$)

Let $\hat{\Sigma} \to \Sigma$ be real oriented blow up of Σ at a_i

(each a_i replaced by a circle ∂_i, so $\partial \hat{\Sigma} = \partial_1 \sqcup \ldots \sqcup \partial_m$)

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(each a_i replaced by a circle ∂_i, so $\partial\hat{\Sigma} = \partial_1 \sqcup \dots \sqcup \partial_m$)

Then each Q_i determines:

1) A connected complex reductive group $H_i \subset G$

2) A finite set $\mathcal{A}_i \subset \partial_i$ of singular directions at a_i

and for each $d \in \mathcal{A}_i$

3) A unipotent group $\mathbb{S}\mathrm{to}_d(Q_i) \subset G$ normalised by H_i

1) H_i = stabilizer of Q_i under adjoint action

$$\left(H_i = \{ g \in G \mid \mathrm{Ad}_g(A_i) = A_i \ \ \forall i \} \right)$$

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so $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid [Y, X] = \alpha(Y) X \ \ \forall Y \in \mathfrak{t} \}$

Let $q_\alpha = \alpha \circ Q$ (mero. function near $a \in \Sigma$)

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Let $q_\alpha = \alpha \circ Q$ (mero. function near $a \in \Sigma$)

then $d \in \partial$ is a singular direction supported by $\alpha \in R$

if $\exp(q_\alpha)$ has maximal decay as $z \to a$ along d

(leading term of q_α is real and negative along d)

& $\mathbb{A} \subset \partial$ is set of all sing. directions $(\forall \alpha \in R)$

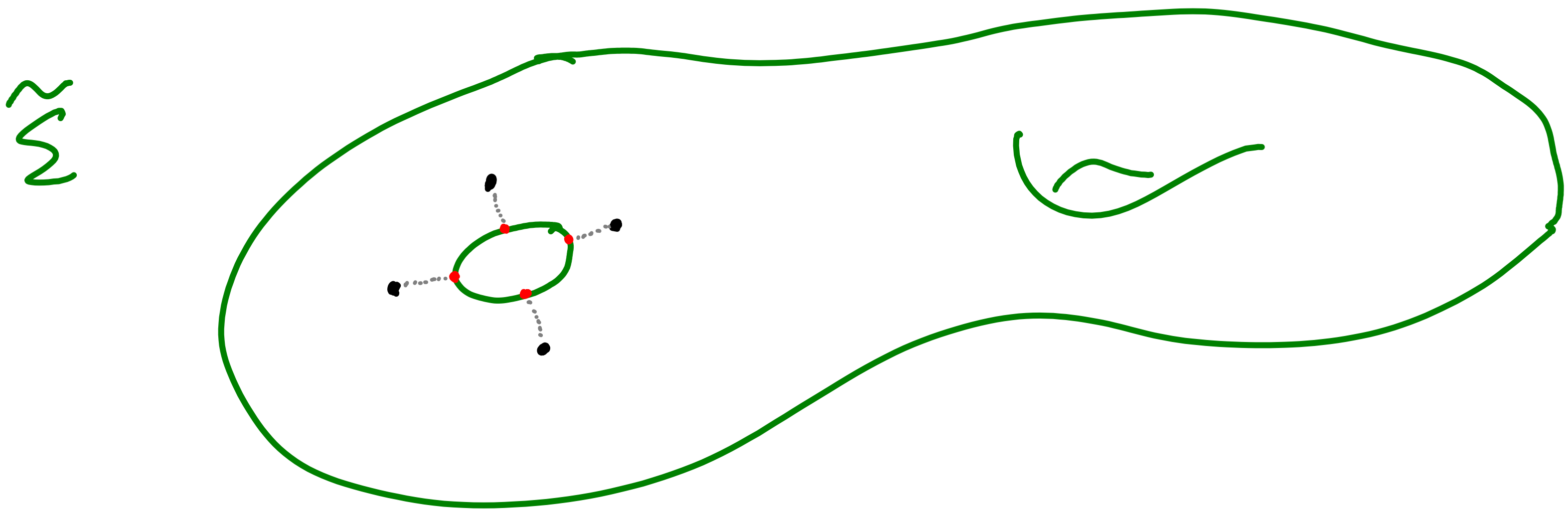
3) Let $R(d) = \{\alpha \mid \alpha \text{ supports } d\} \subset R$

$$St_d = \prod_{\alpha \in R(d)} \exp(\mathfrak{g}_\alpha) \hookrightarrow G$$

Lemma St_d is a well defined unipotent subgroup of G

Now puncture $\hat{\Sigma}$ once in its interior near each singular direction $d \in \mathbb{A}_i$, $i=1,\ldots,m$

and let $\tilde{\Sigma} \subset \hat{\Sigma}$ be resulting punctured surface



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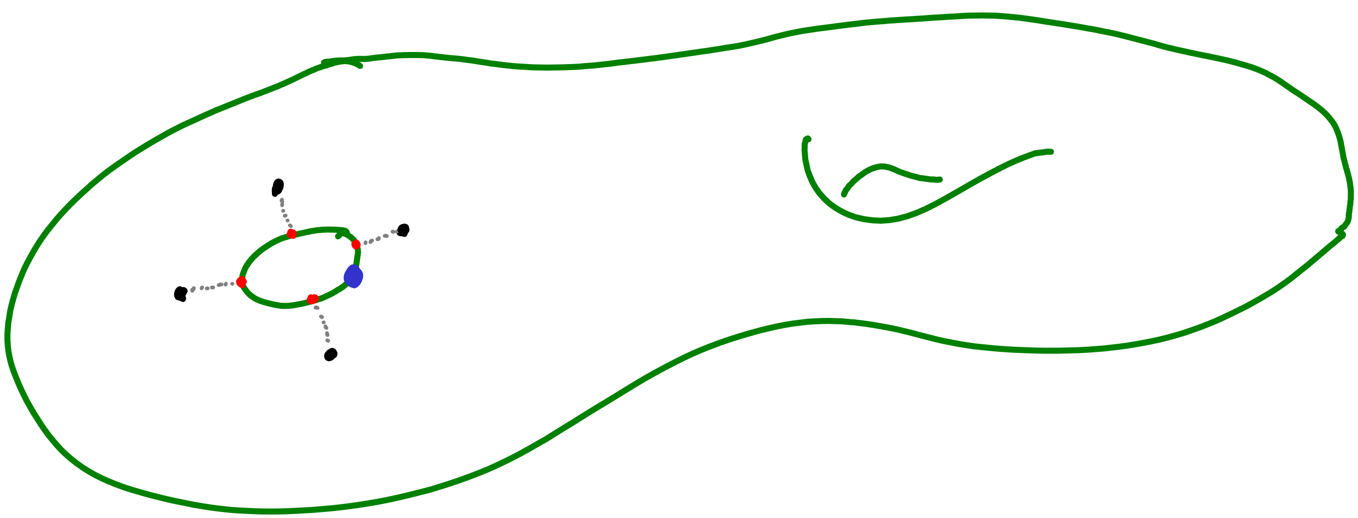

Choose a base point $b_i \in \partial_i$ in each boundary circle

Let $\Pi = \Pi_1(\tilde{\Sigma}, \{b_1, \ldots, b_m\})$

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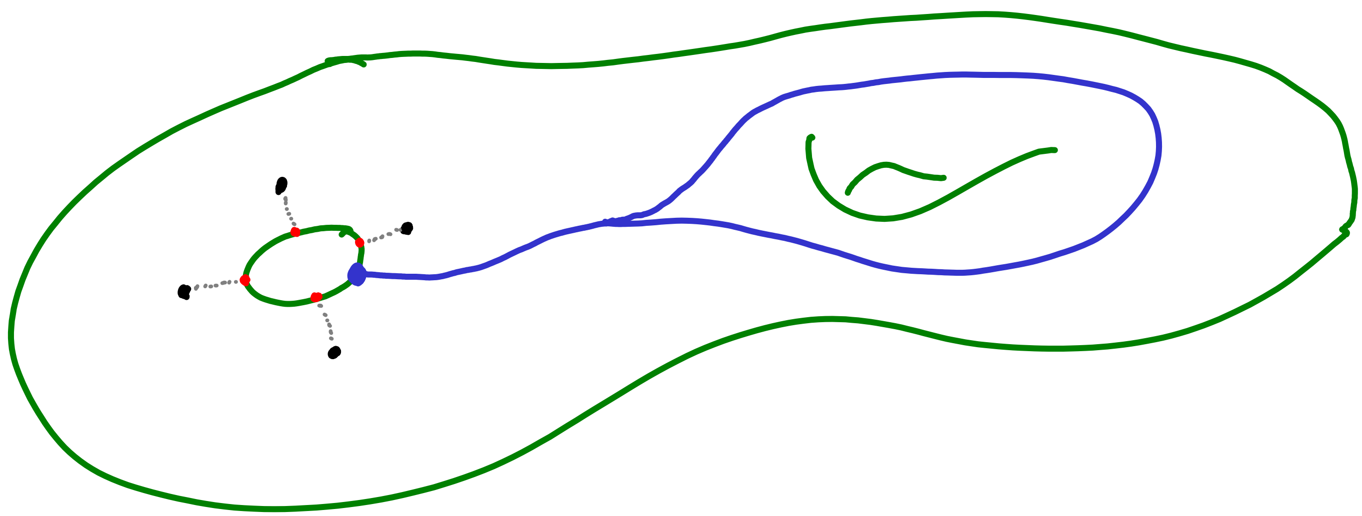

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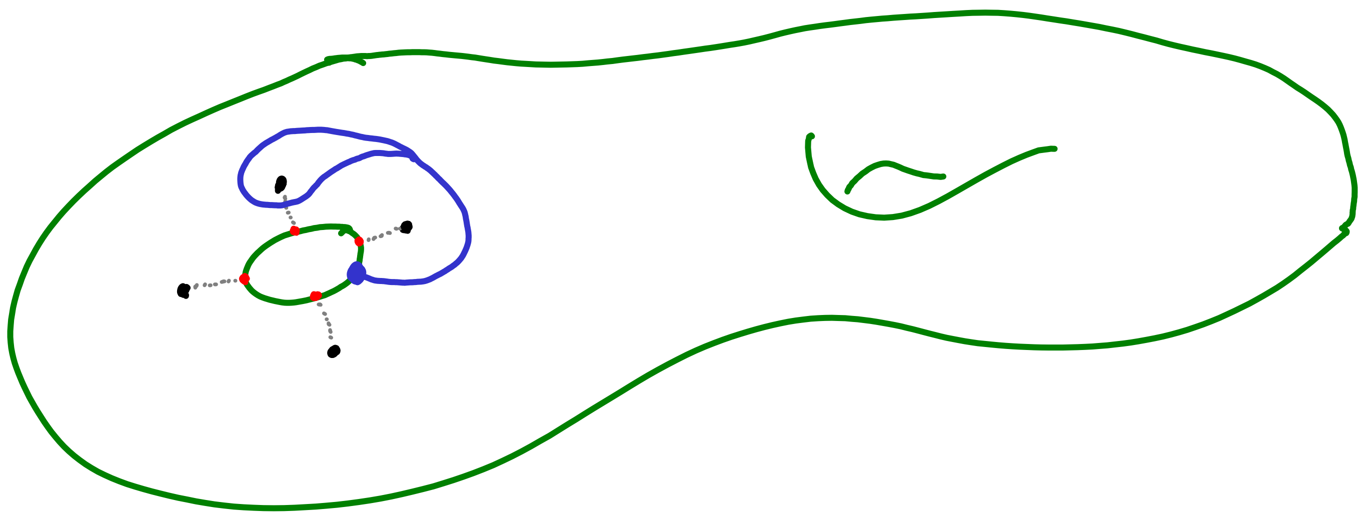

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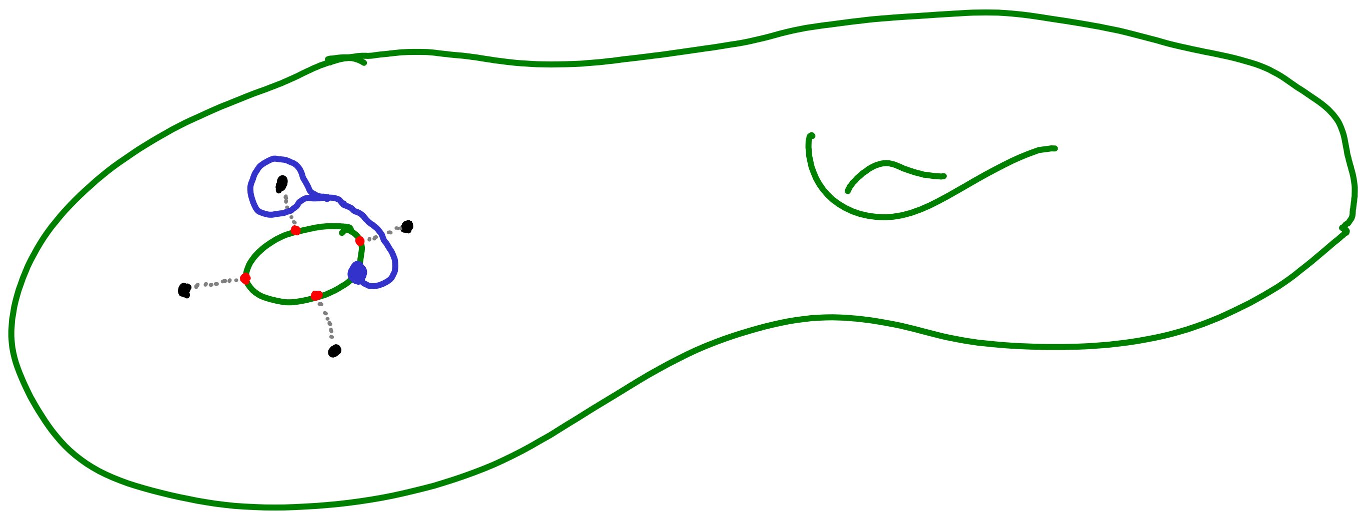

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$$\cup$$

and the subset $\mathrm{Hom}_{\mathbb{S}}(\pi, G)$ of "Stokes representations"

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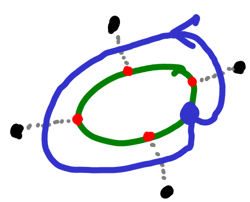

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and the subset $\mathrm{Hom}_{\mathbb{S}}(\pi, G)$ of "Stokes representations" satisfying:

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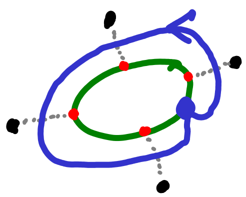
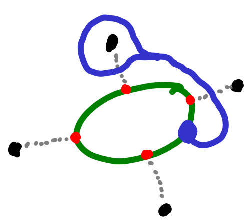


2) If γ goes around ∂_i from b_i until $d \in \mathbb{A}_i$ then loops around the corresponding puncture before returning to b_i, then $\rho(\gamma) \in \mathbb{S}\mathrm{to}_d$



Thm (– '11)

The space of Stokes representations $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ is a smooth affine variety and is (naturally) a quasi-Hamiltonian $\underset{\sim}{H}$-space ($\underset{\sim}{H} = H_1 \times \cdots \times H_m$)

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Corollary

$$M_B(\Sigma, G) := \mathrm{Hom}_{\mathbb{S}}(\Pi, G) / \underset{\sim}{H}$$

inherits an intrinsic Poisson structure (algebraically) with symplectic leaves $\mu^{-1}(e)/\underset{\sim}{H}$ for $e = (e_1, \ldots, e_m) \subset \underset{\sim}{H}$

[M_B classifies irreg. connections with the given irreg. types & Betti weights zero (else use $\hat{e}$)]

Also studied stability for $\underset{\sim}{H} \circlearrowright \mathrm{Hom}_g(\Pi, G)$:

Hilbert-Mumford + general quasi-Hamiltonian properties $\Longrightarrow$

Thm If $\mathcal{C}$ sufficiently generic semisimple conjugacy class

then $\mu^{-1}(\mathcal{C}) / \underset{\sim}{H}$ algebraic symplectic orbifold

(smooth symplectic algebraic variety if $G = GL_n(\mathbb{C})$)

Wild character varieties

(G = connected complex reductive gp)

$$\Sigma \longmapsto \mathrm{Hom}_{\mathbb{S}}(\Pi, G)/\underset{\sim}{H}$$

Irregular curve

Poisson variety

Def^n

A family $\Sigma \to \mathbb{B}$ of irregular curves $(\Sigma_p, a_i, Q_i)_{p \in \mathbb{B}}$ is "admissible" if

1) The fibres Σ_p remain smooth

2) None of the marked points a_i coalesce

3) For each root $\alpha \in R$

$$\mathrm{PoleOrder}\left(\alpha \circ Q_i\right) \in \mathbb{Z}_{\geq 0}$$

is a constant function on $\mathbb{B}$

Thm

If $\Sigma \longrightarrow \mathbb{B}$ is an admissible family of irregular curves

$\Sigma_p = \pi^{-1}(p)$, $p \in \mathbb{B}$

get algebraic Poisson action

$$\pi_1(\mathbb{B}, p) \circlearrowleft \mathrm{Hom}_{\mathbb{S}}(\Pi(p), G)/\underset{\sim}{H}$$

"The Betti moduli spaces $M_B(\Sigma_p, G)$ form a local system of (Poisson) varieties"

E.g. 1) $G = \mathbb{C}^*$ all deformations admissible

Baker functions are hoz. sections of such irreg. connections (on spectral curve)

$\Big\downarrow$ Krichever

Solutions of KdV etc

"times of KdV hierarchy $\longleftrightarrow$ irreg. deformation parameters"

$$\exp\left(xw + t_2 w^2 + t_3 w^3 + \cdots\right) \qquad \begin{cases} w = 1/z \\ x = t_1 \end{cases}$$

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$$\exp\left(xw + t_2 w^2 + t_3 w^3 + \cdots\right) \qquad \begin{cases} w = 1/z \\ x = t_1 \end{cases}$$

2) $Q = \dfrac{-A_0}{z}$, $A_0 \in \mathfrak{t}_{reg}$ $\qquad \left(dQ = \dfrac{A_0}{z^2} dz \right)$

$$\pi_1(\mathfrak{t}_{reg}) = \text{Pure } \mathfrak{g}\text{-braid group}$$

Definition

A "fission variety" is a symplectic or quasi-Hamiltonian variety obtained via the fusion & reduction operations on spaces of the form

1) Conjugacy classes $\mathcal{C} \subset G$ in arbitrary complex reductive groups
2) Fission spaces ${}_G\mathcal{A}_H^r$

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1) Conjugacy classes $\mathcal{C} \subset G$ in arbitrary complex reductive groups
2) Fission spaces ${}_G\mathcal{A}^r_H$

Thm

If Σ is an irregular curve then $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ is a fission variety

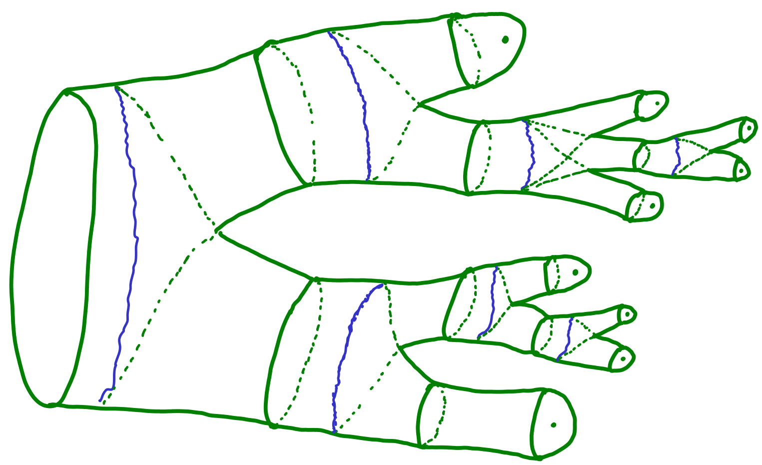
If $Q = \dfrac{A_r}{z^r} + \cdots + \dfrac{A_1}{z}$

Define $G = H_r \supset H_{r-1} \supset \cdots \supset H_0 = H \supset T$

via $H_{i-1} = C_{H_i}(A_i)$

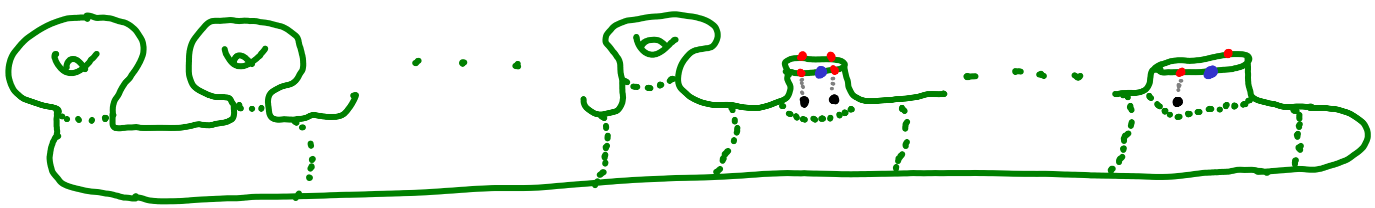
Then $\mathcal{A}(Q) := G \times \{\text{Stokes data for } Q\} \times H$ obtained by gluing

$$\mathcal{A}(Q) \cong {}_G\mathcal{A}^r_{H_{r-1}} \circledast_{H_{r-1}} \mathcal{A}^{r-1}_{H_{r-2}} \circledast \cdots \circledast_{H_1} \mathcal{A}^1_H$$



If Σ an irregular curve :

$$\mathrm{Hom}_{\mathbb{S}}(\Pi, G) \cong \underbrace{\mathbb{D} \circledast \cdots \circledast \mathbb{D}}_{g} \circledast \mathcal{A}(Q_1) \circledast \cdots \circledast \mathcal{A}(Q_m) /\!/ G$$



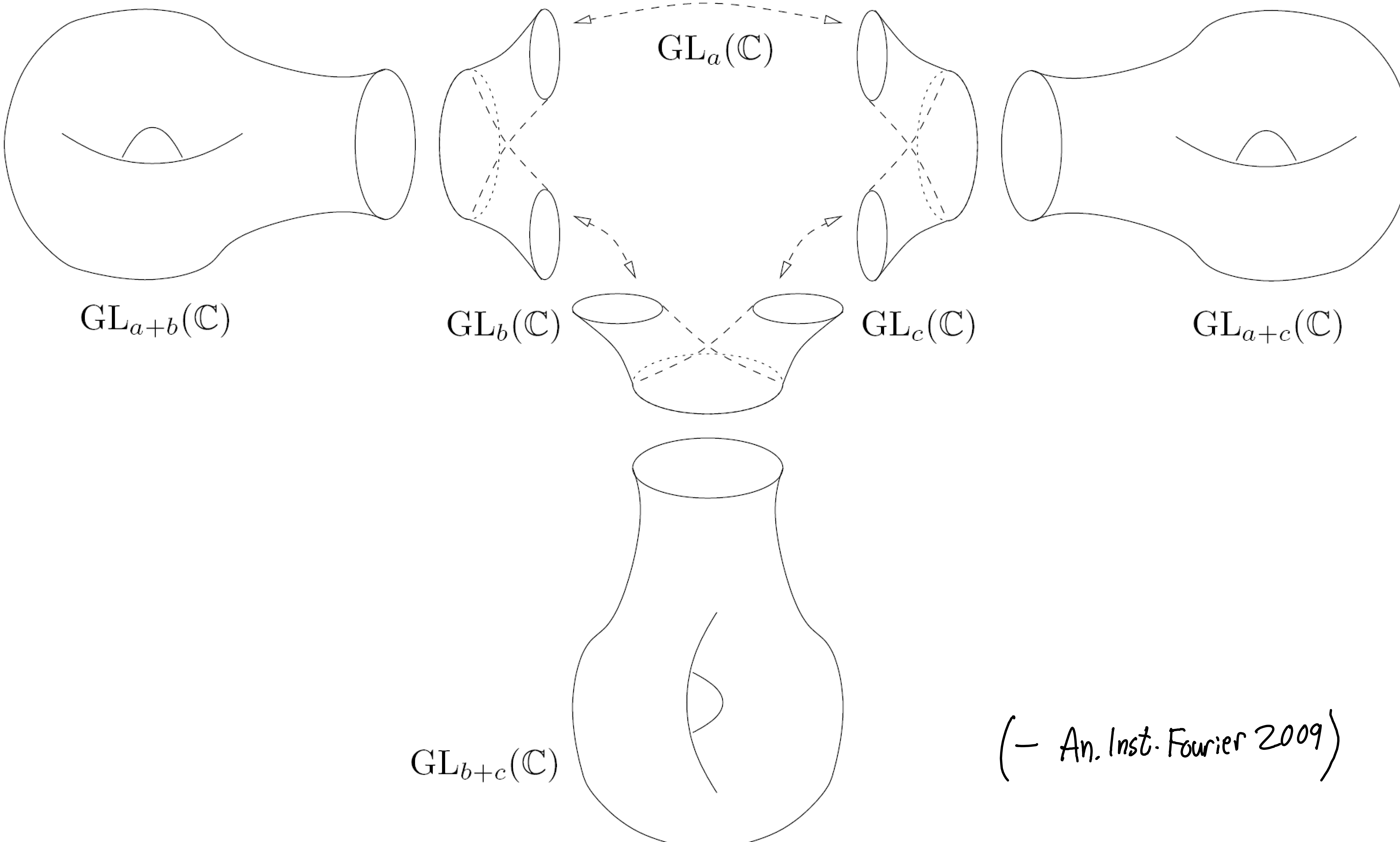
$$\mu^{-1}(e)/\underset{\sim}{H} \cong \left\{ (\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \underset{\sim}{h}, \underset{\sim}{S}) \,\middle|\, \prod_{i=1}^{g} [A_i, B_i] \prod_{i=1}^{m} \mu_i = 1,\ h_i^{-1} \in \mathcal{C}_i \right\} / \underset{\sim}{H}$$

$$\mu_i = C_i^{-1} h_i \cdots S_2^{(i)} S_1^{(i)} C_i$$

But there are many other examples of fission varieties

- e.g. can glue surfaces $\widetilde{\Sigma}$ along their boundaries (provided the groups H_i match up)

- can obtain all the so-called multiplicative quiver varieties



(— An. Inst. Fourier 2009)

The amazingly rich structure of the solutions of the first Painlevé equation, the very simple differential equation $d^2y/dx^2 = 6y^2 + x$.

Hans Duistermaat

Abstract: The differential equation $d^2y/dx^2 = 6y^2 + x$ is the first in Painlevé's classification of those algebraic second order ordinary differential equations such that every isolated singularity of moderate growth of every solution is a pole. Despite the simple form of the first Painlevé equation, the analysis of its solutions is very subtle. Painlevé thought in 1900 to have proved that every solution can be extended to a meromorphic function on the whole complex *x*--plane, but the first complete proofs were only given almost a century later. Boutroux discovered in 1913 a transformation to an approximately autonomous differential equation, which allowed him to draw spectacular conclusions about the asymptotic behavior of the locations of the poles near infinity in the complex plane. His proofs contained serious gaps and errors, but his conclusions were basically correct, and can be strengthened considerably. In this talk I would like to report on my joint work in progress with Nalini Joshi on this.

The first Painlevé equation can also be obtained as a compatibility condition between two linear systems of ordinary differential equations, one in the variable *x* with a parameter λ, and the other with the roles of *x* and λ interchanged. This has been used to study the asymptotic properties of the first Painlevé transcendants, but our analysis does not use this, in the literature called "isomonodromy method".