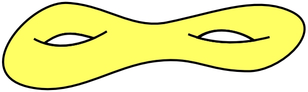
Moduli spaces of flat connections on colored surfaces

David Li-Bland
joint work with Pavol Ševera

Tuesday, July 31, 2012

Moduli space of flat connections

Let $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ be a quadratic Lie algebra,

$\Sigma =$ 

Theorem (Atiyah-Bott)

The moduli space

$$\mathcal{M}(\Sigma) = \mathcal{A}_{flat}(\Sigma)/C^{\infty}(\Sigma, G)$$

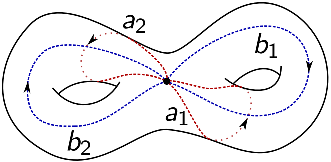
of flat connections over Σ carries a symplectic structure.

Proof.

Infinite dimensional symplectic reduction... $\square$

Finite Dimensional Construction

Triangulate the surface:



A flat connection assigns an element of G (the holonomy) to each edge. $\mathcal{M}(\Sigma)$ is collection of possible (coherent) assignments.

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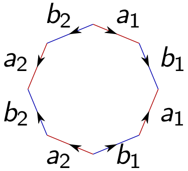

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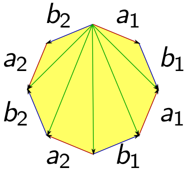

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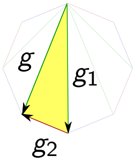

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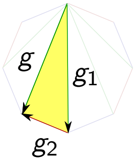

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Flat connections and triangulations

A triangulation breaks our surface into

- vertices (0-dimensional simplex)
- edges (1-dimensional simplex)
- faces (2-dimensional simplex)

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Flat connections on the 1-simplex

- $\mathcal{A}_{flat}([0,1]) :=$ flat connections over $[0,1]$
- $C^{\infty}_{based}([0,1], G) := \{f \text{ such that } f(0) = f(1) = \mathrm{id}\}$

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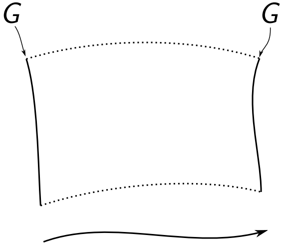

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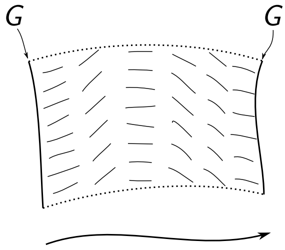

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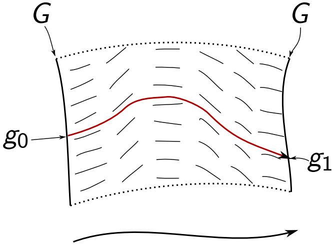

Holonomy $= g_0^{-1} g_1$

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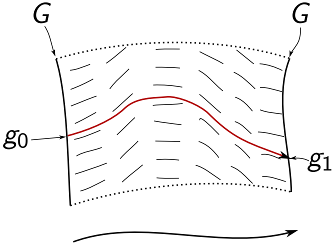

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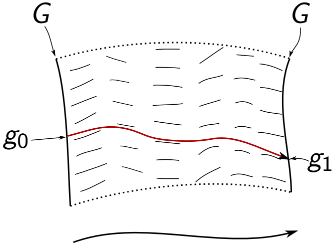

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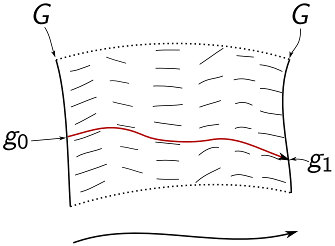

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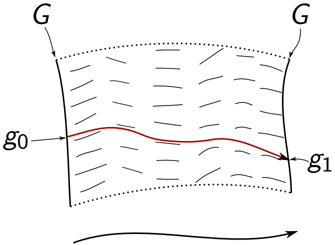
$$\mathrm{hol} : \mathcal{A}_{flat}([0,1]) / C^\infty_{based}([0,1],G) \xrightarrow{\cong} G$$

Residual gauge transformations



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$$\begin{tikzcd} \mathcal{A}_{flat}([0,1]) \arrow[loop above, "C^\infty([0,1],G)"] \end{tikzcd}$$

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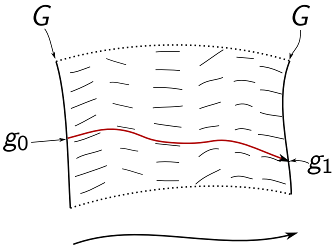

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$$
\begin{tikzcd}
C^\infty([0,1], G)/C^\infty_{based}([0,1], G) \\
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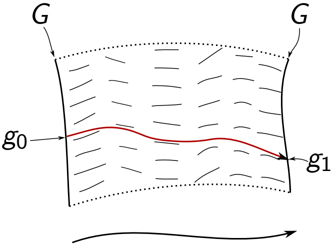

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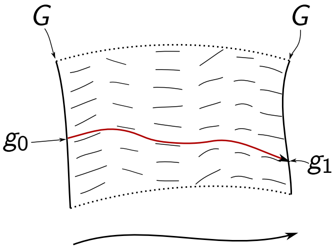

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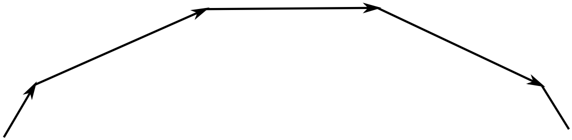
Pictoral Notation



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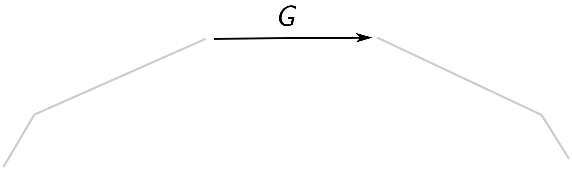
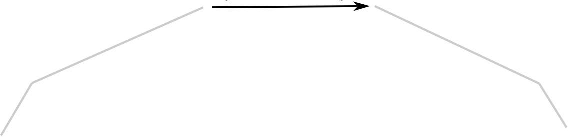


Pictoral Notation

$$\bar{\mathfrak{g}} \circlearrowleft G \circlearrowright \mathfrak{g}$$



Flat connections on the 2-simplex

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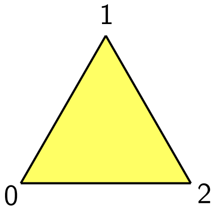

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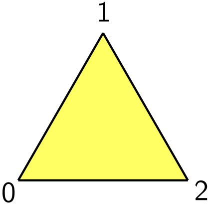

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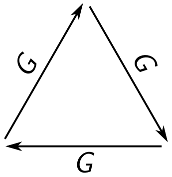

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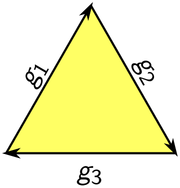

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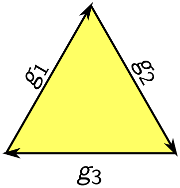

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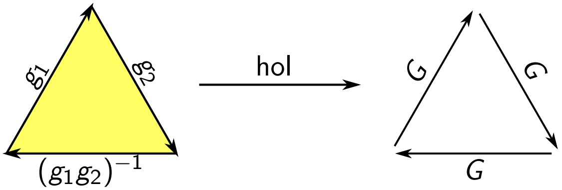

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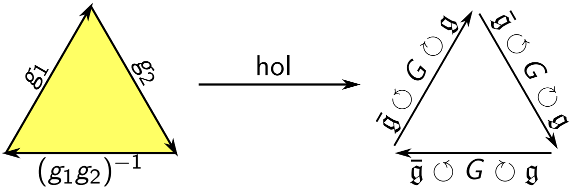

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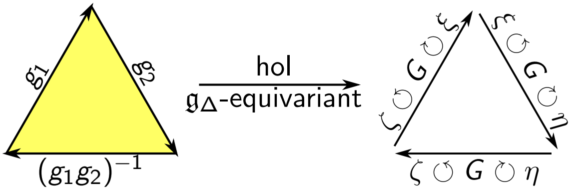

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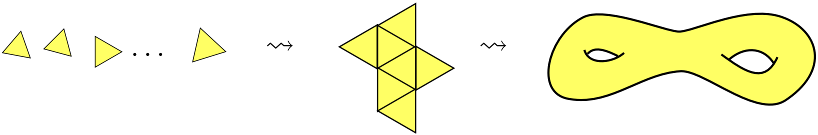
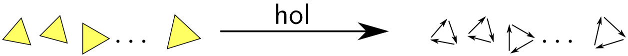


2-simplices



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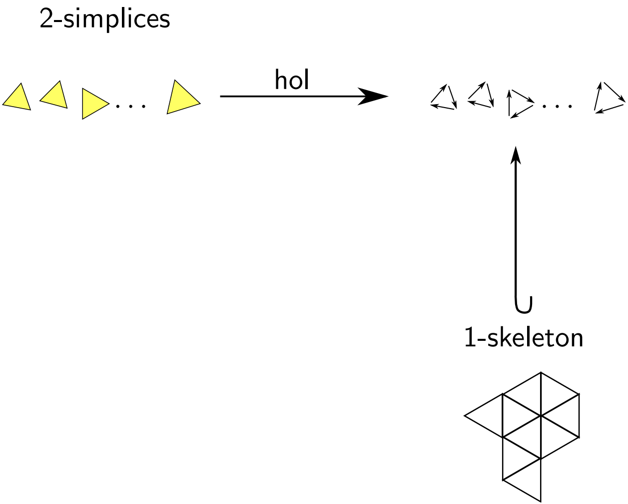

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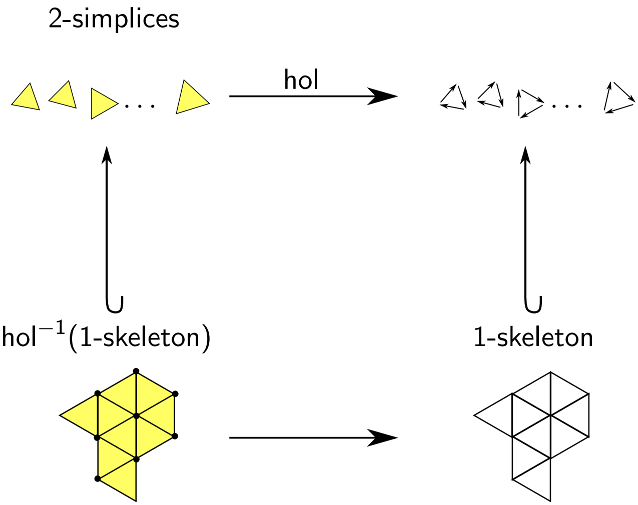

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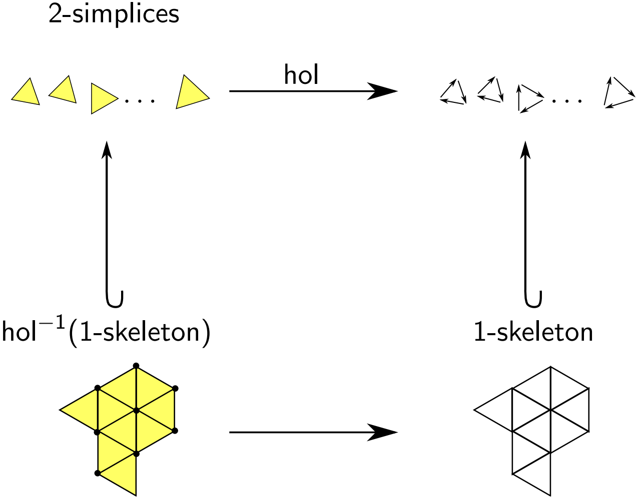

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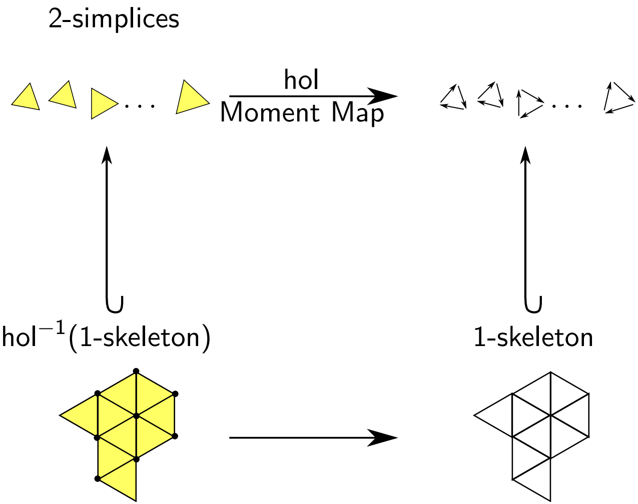

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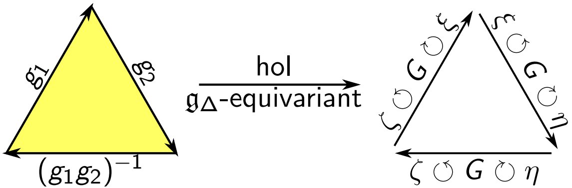
Moment Map (Bursztyn,Iglesias-Ponte,Ševera)



- $(\bar{\mathfrak{g}} \oplus \mathfrak{g}) \times G$ is a Courant algebroid.
- $\mathfrak{g}_\Delta \subseteq (\bar{\mathfrak{g}} \oplus \mathfrak{g})^3$ defines a Dirac structure.
- The action of $\mathfrak{g}_\Delta$ on $\mathcal{M}(\Delta)$ is Hamiltonian for the unique 2-form
$$\omega = \frac{1}{2}\langle g_1^{-1} dg_1, dg_2 g_2^{-1}\rangle \in \Omega^2(\mathcal{M}(\Delta)).$$

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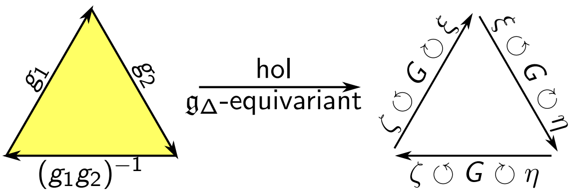

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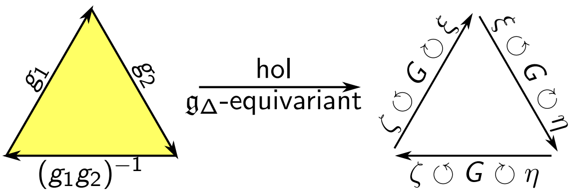

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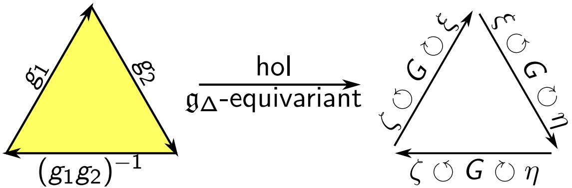

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Reduction, part1

$$\begin{tikzcd} \mathcal{M}(\Delta)^n \arrow[r, "\mathrm{hol}"] & G^{3n} \\ \mathfrak{g}^n_\Delta \arrow[u, "\circlearrowleft"] & \mathfrak{g}^n_\Delta \subseteq (\bar{\mathfrak{g}} \oplus \mathfrak{g})^{3n} \arrow[u, "\circlearrowleft"] \end{tikzcd}$$

(moment map = holonomy)

Choose $\mathfrak{l} \subseteq (\bar{\mathfrak{g}} \oplus \mathfrak{g})^{3n}$.

$$\begin{tikzcd} \mathrm{hol}^{-1}(\mathfrak{l} \cdot \mathrm{id}) \arrow[r, "\mathrm{hol}"] & \mathfrak{l} \cdot \mathrm{id} \\ \mathfrak{l} \cap \mathfrak{g}^n_\Delta \arrow[u, "\circlearrowleft"] & \mathfrak{l} \cap \mathfrak{g}^n_\Delta \arrow[u, "\circlearrowleft"] \end{tikzcd}$$

(moment level $\mathfrak{l} \cdot \mathrm{id} \subseteq G^{3n}$)

Reduction, part1

$$\begin{tikzcd} \mathcal{M}(\Delta)^n \arrow[rr, "\mathrm{hol}"] \arrow[loop below] & & G^{3n} \arrow[loop below] \\ \mathfrak{g}_\Delta^n & & \mathfrak{g}_\Delta^n \subseteq (\bar{\mathfrak{g}} \oplus \mathfrak{g})^{3n} \end{tikzcd} \qquad (\text{moment map} = \text{holonomy})$$

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Reduction, part2

Theorem (Li-Bland, Ševera)

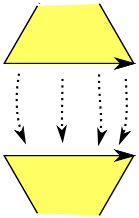
Suppose $\mathfrak{l} \subseteq (\bar{\mathfrak{g}} \oplus \mathfrak{g})^{3n}$ *is a Lagrangian Lie subalgebra.*
Then, under suitable transversality assumptions, the restriction of the 2-form to

$$\mathrm{hol}^{-1}(\mathfrak{l} \cdot \mathrm{id}) \subseteq \mathcal{M}(\Delta)^n$$

descends to define a symplectic form on

$$\mathrm{hol}^{-1}(\mathfrak{l} \cdot \mathrm{id})/(\mathfrak{g}^n_\Delta \cap \mathfrak{l}).$$

Sewing edges together



$$\mathfrak{l} \subseteq (\bar{\mathfrak{g}} \oplus \mathfrak{g})^2$$

$\mathfrak{l} \cdot \mathrm{id}$

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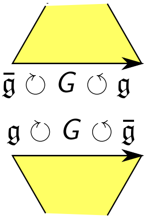

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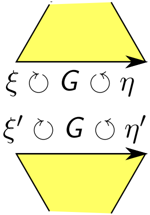
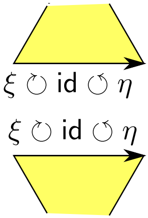


$$\mathfrak{l} := \{\xi = \xi' \text{ and } \eta = \eta'\} \subseteq (\bar{\mathfrak{g}} \oplus \mathfrak{g})^2$$

$\mathfrak{l} \cdot \mathrm{id}$

Sewing edges together



$$\mathfrak{l} := \{\xi = \xi' \text{ and } \eta = \eta'\} \subseteq (\bar{\mathfrak{g}} \oplus \mathfrak{g})^2$$

$$\mathfrak{l} \cdot \mathrm{id} \ = ?$$

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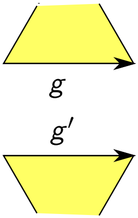

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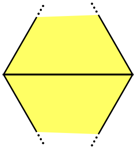

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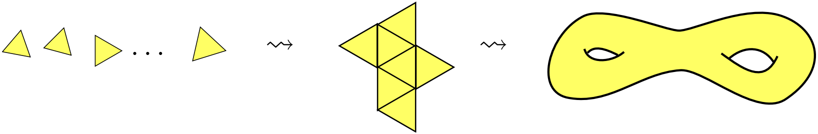
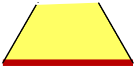


Coloring edges

Suppose $\mathfrak{c} \subseteq \mathfrak{g}$ is coisotropic ($\mathfrak{c}^{\perp} \subseteq \mathfrak{c}$).



$$\mathfrak{l} \subseteq (\bar{\mathfrak{g}} \oplus \mathfrak{g})$$

$$\mathfrak{l} \cdot \mathrm{id} := C^{\perp}$$

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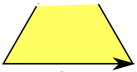

$$\bar{\mathfrak{g}} \circlearrowright G \circlearrowleft \mathfrak{g}$$

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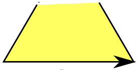

$\xi \circlearrowright G \circlearrowleft \eta$

$$\mathfrak{l} := \{\xi, \eta \in \mathfrak{c} \text{ and } \xi - \eta \in \mathfrak{c}^\perp\} \subseteq (\bar{\mathfrak{g}} \oplus \mathfrak{g})$$

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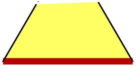

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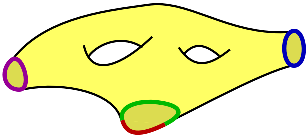
Surfaces with colored boundaries



$\mathfrak{c}_1$

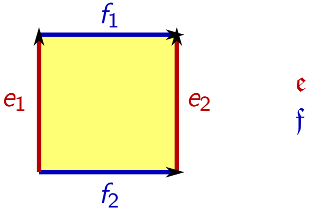
$\mathfrak{c}_2$

$\mathfrak{c}_3$

$\mathfrak{c}_4$

Symplectic double groupoid (Ševera)

Suppose that $\mathfrak{e}, \mathfrak{f} \subseteq \mathfrak{g}$ are two transverse Lagrangian subalgebras, then the moduli space

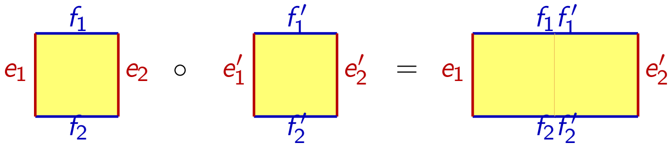


$$\mathcal{M} = \{(e_1, e_2, f_1, f_2) \in E^2 \times F^2 \mid e_1 f_1 = f_2 e_2\}$$

$$\Omega = \langle e_1^{-1} de_1, df_1 f_1^{-1} \rangle - \langle f_2^{-1} df_2, de_2 e_2^{-1} \rangle$$

is the symplectic double groupoid associated to the Manin triple $(\mathfrak{g}, \mathfrak{e}, \mathfrak{f})$.

Multiplication

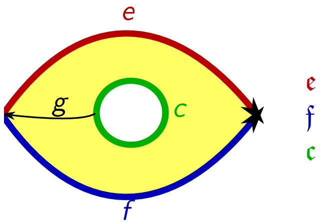


Composable elements satisfy

$$e_2 = e_1'.$$

Symplectic groupoid for Lu-Yakimov Poisson structures

Suppose $(\mathfrak{g}, \mathfrak{e}, \mathfrak{f})$ is a Manin triple and $\mathfrak{c} \subseteq \mathfrak{g}$ is coisotropic.



$$\mathcal{M} = \{(c, g, e, f) \in C^{\perp} \times_C G \times E \times F \text{ such that } cgef^{-1}g^{-1} = \text{id}\}$$

This is the symplectic groupoid for the Lu-Yakimov Poisson structure on G/C.

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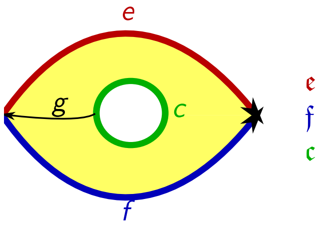

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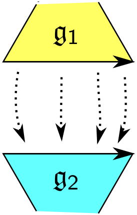
Domain Walls

Let $(\mathfrak{g}_i, \langle\cdot,\cdot\rangle_i)$ be two quadratic Lie algebras. Suppose that $\mathfrak{c} \subseteq \mathfrak{g}_2 \oplus \bar{\mathfrak{g}}_1$ is a coisotropic subalgebra.



$$\mathfrak{l} := \{(\xi', \xi), (\eta', \eta) \in \mathfrak{c} \text{ and } (\xi', \xi) - (\eta', \eta) \in \mathfrak{c}^{\perp}\}$$

- $\mathfrak{g}_1 = \mathfrak{g}_2$, and $\mathfrak{c} = \{(\xi, \xi)\} \Leftrightarrow$ sewing the edges together.
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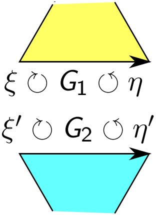

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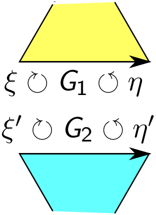

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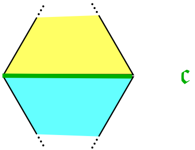

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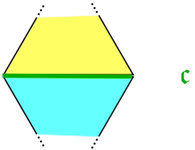

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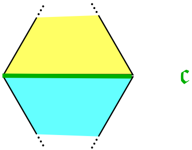

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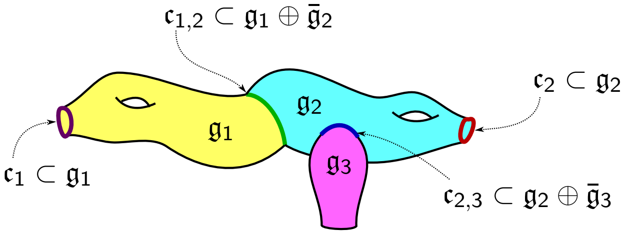
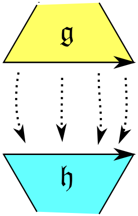
Colored surfaces with domains



Example (Philip Boalch)

Suppose $\mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{u}_+ \oplus \mathfrak{h}$ as a vector space, where $\mathfrak{p}_\pm := \mathfrak{u}_\pm \oplus \mathfrak{h} \subseteq \mathfrak{g}$ is a coisotropic subalgebra with $\mathfrak{p}_\pm^\perp = \mathfrak{u}_\pm$.



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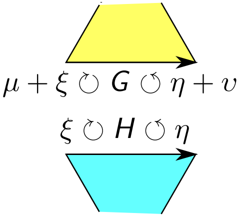

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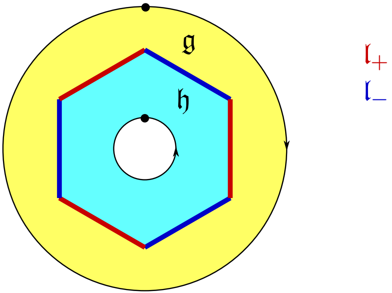


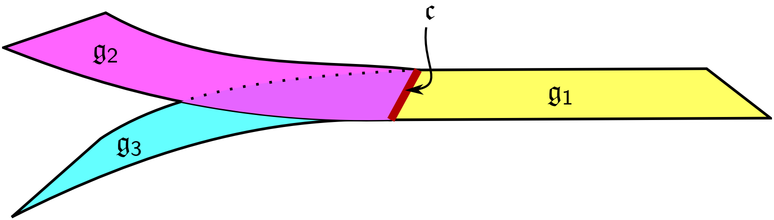
This moduli space is Philip Boalch's fission space

$$ {}_H\mathcal{A}^r_G := G \times (U_+ \times U_-)^r \times H, $$

(for $r = 3$).

Coloring n edges

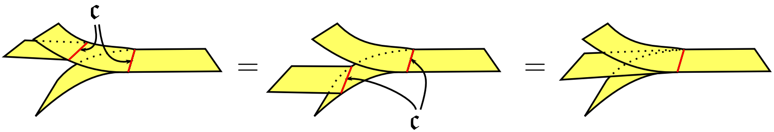
Suppose $\mathfrak{c} \subseteq \oplus_{i=1}^{n} \mathfrak{g}_i$ is coisotropic.



$$\mathfrak{l} = \Big\{ \big((\xi_1, \eta_1); \ldots; (\xi_n, \eta_n) \big) \in \bigoplus_{i=1}^{n} \bar{\mathfrak{g}}_i \oplus \mathfrak{g}_i \mid$$

$$(\xi_1, \ldots, \xi_n), (\eta_1, \ldots, \eta_n) \in \mathfrak{c} \text{ and } (\xi_1 - \eta_1, \ldots, \xi_n - \eta_n) \in \mathfrak{c}^{\perp} \Big\}$$

branched surfaces and quasi-triangular structures



Then $\mathfrak{c}$ defines an (associative) multiplication $\circ : \mathfrak{g} \times \mathfrak{g} \dashrightarrow \mathfrak{g}$:

$$\mathfrak{c} = \{(\xi, \xi', \xi'') \mid \xi \circ \xi' \circ \xi'' = \mathrm{id}\} \subseteq \mathfrak{g} \oplus \overline{\mathfrak{g} \oplus \mathfrak{g}}$$

i.e. $\mathfrak{g} \rightrightarrows \mathfrak{k}$ is a Lie groupoid.

Lemma (Drinfel'd)

Suppose $\mathfrak{k}$ is a Lie algebra, elements $s \in S^2(\mathfrak{k})^{\mathfrak{k}}$ are in one-to-one correspondence with quadratic Lie algebras $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ such that $\mathfrak{g} \rightrightarrows \mathfrak{k}$ is a Lie groupoid. $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is called the double of $(\mathfrak{k}, s)$.

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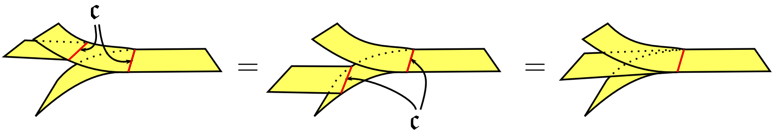

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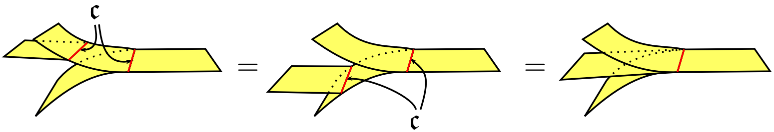

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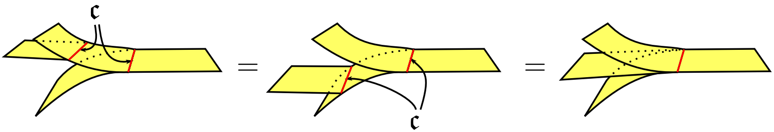

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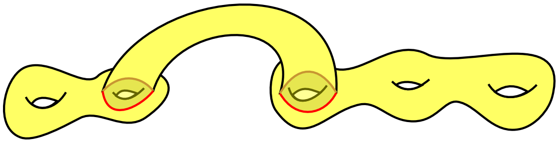
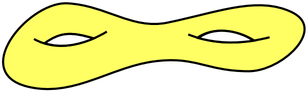


Poisson Structures

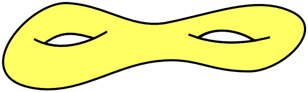
Suppose $\mathfrak{k}$ is a Lie algebra and $s \in S^2(\mathfrak{k})^{\mathfrak{k}}$ is non-degenerate.

$\Sigma =$ 

The moduli space, $\mathcal{M}(\Sigma)$, of flat $\mathfrak{k}$ connections over Σ carries a symplectic structure.

Poisson Structures

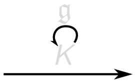
Suppose $\mathfrak{k}$ is a Lie algebra and $s \in S^2(\mathfrak{k})^{\mathfrak{k}}$ is possibly degenerate.

$\Sigma =$ 

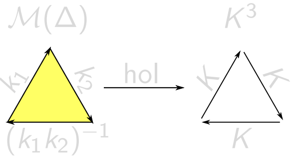
The moduli space, $\mathcal{M}(\Sigma)$, of flat $\mathfrak{k}$ connections over Σ carries a Poisson structure.

Idea

- Recall the Drinfel'd double, $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, of $(\mathfrak{k}, s)$.
- $\mathfrak{g}$ acts on K. Model $\mathcal{M}([0,1])$ by the Courant Algebroid



- Moment map:



- Construct Poisson structure on $\mathcal{M}(\Sigma)$ using Dirac reduction.

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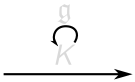

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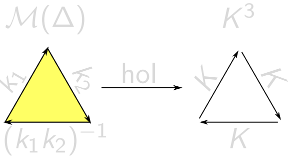

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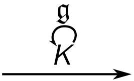

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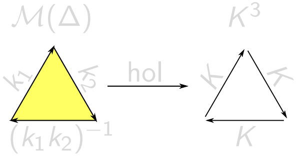
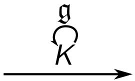


- Construct Poisson structure on $\mathcal{M}(\Sigma)$ using Dirac reduction.

Idea

- Recall the Drinfel'd double, $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, of $(\mathfrak{k}, s)$.
- $\mathfrak{g}$ acts on K. Model $\mathcal{M}([0,1])$ by the Courant Algebroid



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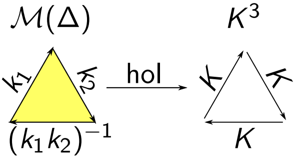

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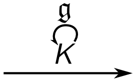

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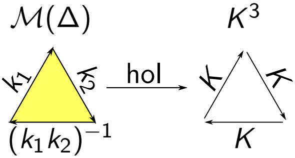

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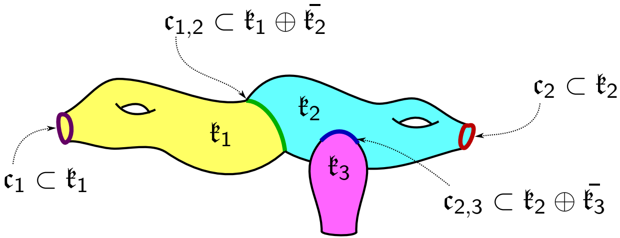
Colored surfaces with domains



Get Poisson structure.