Poisson manifolds of compact type

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Joint work with M. Crainic and R. L. Fernandes

- A **Poisson manifold of compact type (PMCT)** (M, π) is the Lie algebroid of a compact (Hausdorff) symplectic groupoid $(\mathcal{G}, \Omega) \rightrightarrows (M, \pi)$.

- A **Poisson manifold of strong compact type (PMCT)** is the Lie algebroid of a compact (Hausdorff) source 1-connected symplectic groupoid $(\Sigma(M), \Omega) \rightrightarrows (M, \pi)$.

- For an appropriate class of Poisson manifolds, is there a Poisson topology ?

- Focus our attention on strong compact type Poisson manifolds:
  - We do not know of other integrations of $(T^*M, [\cdot,\cdot]_\pi)$; if there are they might not be symplectic.
  - We have an explicit (but complicated) model for $\Sigma(M) \rightrightarrows M$ which will allow characterizing PMCT.
  - We can "exponentiate" constructions from $(T^*M, [\cdot,\cdot]_\pi)$ to $\Sigma(M)$, but not to other integrations $\mathcal{G} \rightrightarrows M$, so we can draw more consequences from CT condition.

Redefine

Poisson manifold of compact type $\cong$ integrable with compact source 1-connected integration

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3. Present a construction of non-trivial (i.e. not symplectic) PMCT related to quasi-Hamiltonian Abelian spaces.

- Integral affine structures are the key tool for global results.

- Generalize (M,π) PMCT in two directions:

1. Properties of $\Sigma(M)$: **PMCT** $\subsetneq$ $\subsetneq$ Proper

   (M,π) PMCT $=$ (M,π) $\underline{s}$-proper $+$ M compact.

Example (PMCT)

(S,ω), S compact symplectic manifold with finite π_1.

Example ($\underline{s}$-proper PM)

$(S\times\mathfrak{g},\omega\times\pi_{\text{lin}})$, S compact symplectic manifold with finite π_1, $\mathfrak{g}$ semisimple of compact type.

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▷ (M, π) $\underline{s}$-proper, $\mathcal{F}$ characteristic foliation:

- Isotropy groups are compact.
- Leaves of $\mathcal{F}$ are compact with finite π_1.
- $M/\mathcal{F}$ is Hausdorff.

  Use

  (i) $G_x \circlearrowright \underline{s}^{-1}(x) \xrightarrow{\underline{t}} F_x$,

  (ii) $\underline{s}^{-1}(x)$ compact and 1-connected.

▷ If (M, π) regular, then $M/\mathcal{F}$ admits orbifold structure (this will be true as well in the non-regular case).

▷ (M^n, π) $\underline{s}$-proper:

- $H^*_\pi(M) = \Gamma(E^*)^{\Sigma(M)}$, $E \to M$ vector bundle, $E_x = H^*(\underline{s}^{-1}(x))$.

  Poisson cohomology complex $\equiv$ right invariant fiberwise (w.r.t. $\underline{s}$) differential forms.

- $H^1_\pi(M) = 0 \Rightarrow (M, \pi)$ unimodular, Poisson actions are Hamiltonian.

- M orientable, Poincaré duality pairing:

  $H^k_\pi(M) \times H^{n-k}_\pi(M) \to E^{k\,\Sigma(M)} \times E^{n-k\,\Sigma(M)} \xrightarrow{\int_{\underline{s}^{-1}(x)}} C^\infty(M/\mathcal{F}) \xrightarrow{\int_{M/\mathcal{F}}} \mathbb{R}$

  M compact (PMCT), μ Hamiltonian invariant volume form,

$$(P, Q) \mapsto n \int_M (i_{P\wedge Q}\mu)\mu,\ n \in \mathbb{N}^*,$$

  non-degenerate pairing.

- M orientable, $H^k_\pi(M) \cong H^{n-k}_\pi(M) \Rightarrow H^{n-1}_\pi(M) = 0$.

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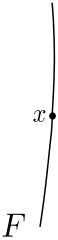
- Linearization around F leaf of (M, π), $\underline{s}$-proper [Zung]:



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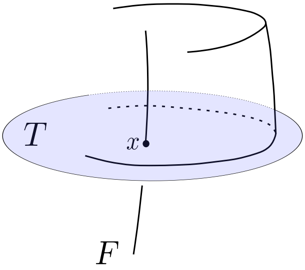

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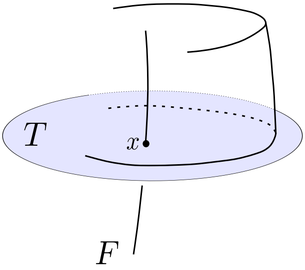

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$$(\Sigma(M)_T, \Omega) \cong (G \ltimes \mathfrak{g}^*, d\lambda)_U$$

$$\underline{s}\colon (\underline{s}^{-1}(T), \Omega) \to U \subset \mathfrak{g}^*$$

free G*–Hamiltonian space*

- $(V_T, \pi) \cong (s^{-1}(T), \Omega)/G$.
- $(\Sigma(M)_{V_T}, \Omega) \cong (s^{-1}(T) \times s^{-1}(T), \Omega \oplus -\Omega)//G$.

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- There exist free compact Abelian quasi-Hamiltonian spaces [McDuff, Kotschick].
- There are no free compact non-Abelian quasi-Hamiltonian spaces [Alekseev-Meinrenken-Woodward].

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1. $(V_T, \pi) \cong (s^{-1}(T), \Omega)/G$: Normal form for Hamiltonian G-spaces [Guillemin-Sternberg].
   (i) $P = \underline{s}^{-1}(x) \to F$ 1-connected principal G-bundle;
   (ii) (F, ω),
   $\underline{s}\colon (s^{-1}(T), \Omega) \to T \cong \pi_2\colon (P \times \mathfrak{g}^*, \omega + d\langle \eta, \xi \rangle)_{V_T} \to U \subset \mathfrak{g}^*$, η connection 1-form on P, ξ coordinate on $\mathfrak{g}^*$ (same normal form as in [Crainic-Markut]).

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2. $(\Sigma(M)_{V_T}, \Omega) \cong (s^{-1}(T) \times s^{-1}(T), \Omega \oplus -\Omega)//G$: If (Y, σ, μ) compact free quasi-Hamiltonian G-space,

$$(Y, \sigma, \mu) \circledast (Y, -\sigma, \mu^{-1})//G \rightrightarrows (Y/G, \pi_{\mathrm{red}}).$$

- There exist free compact Abelian quasi-Hamiltonian spaces [McDuff, Kotschick].
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- Dirac case (M, L):
  - Cohomology, Poincaré duality pairing, etc, is about an integrable Lie algebroid A whose source 1-connected integration is $\underline{s}$-proper (resp. compact).
  - Same normal form around presymplectic leaf (F, ω) with isotropy group G, using extended symmetries

$$\pi_2 \colon (P \times \mathfrak{g}^*, \omega + d\langle \eta, \xi \rangle + \beta) \to \mathfrak{g}^*,$$

$$(V_T, L) \cong (P \times U/G, L_{\mathrm{red}} + d\gamma), \;\; U \subset \mathfrak{g}^*.$$

▷ An **integral affine structure (i.a.s.)** on a manifold B^d is given by an atlas $\{(U_i, \phi_i)\}_{i \in I}$ with change of coordinates

$$\phi_{ij} \in \mathrm{Aff}(\mathbb{R}^d, \mathbb{Z}^d) := \mathbb{R}^d \rtimes \mathrm{GL}(\mathbb{Z}^d).$$

▷ Four facts about i.a. structures:

1. An i.a.s. is equivalently given by $\Lambda \subset T^*M$ such that Λ_b is a full rank lattice and Λ is locally given by closed forms.

   I. a. coordinates $\lambda_1, \dots, \lambda_d \equiv \Lambda$ generated by $d\lambda_1, \dots, d\lambda_d$.

   Dual description $\check{\Lambda} \subset TM$ locally by pairwise commuting fields.

2. $(X, \sigma) \to B$ Lagrangian fibration with compact connected fibers induces (B, Λ) i.a. structure [Duistermaat].

3. I.a. manifolds have **developing maps**

$$D\colon (\tilde{B},\tilde{\Lambda}) \to (\mathbb{R}^d,\mathbb{Z}^d),$$

with **affine holonomy representation**

$$\rho\colon \pi_1(B,b) \twoheadrightarrow \Gamma \subset \mathrm{Aff}(\mathbb{R}^d,\mathbb{Z}^d),$$

▷ (B,Λ) is **complete** if D is a diffeomorphism. In such case ρ is faithful and

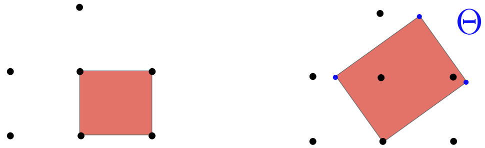
$$(B,\Lambda) \cong (\mathbb{R}^d,\mathbb{Z}^d)/\rho(\pi_1(B,b))$$

Conjecture (Markus)

All compact i.a. manifolds are complete.

Example

$(\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d, \mathbb{Z}^d)$ is an i.a.manifold. More generally for $\Theta \subset \mathbb{R}^d$ any full rank lattice, $(\mathbb{R}^d/\Theta, \mathbb{Z}^d)$ is an i.a. structure on the torus.



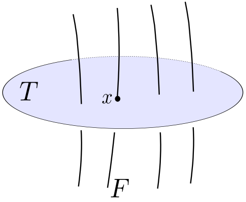
4. On (B, Λ) i.a. manifold one can make sense of polynomials.

   On $(\mathbb{T}^1, \mathbb{Z})$ a polynomial f is given by $\tilde{f}: \mathbb{R} \to \mathbb{R}$ 1-periodic polynomial $\Rightarrow f$ must be constant!.

   ▷ A compact i.a. manifold only supports constant polynomials [Goldman-Hirsch].

(M, π) regular $\underline{s}$-proper, leaves of $\mathcal{F}$ 1-connected, $M \to B = M/\mathcal{F}$.

▷ B is an i.a. manifold [Zung]:

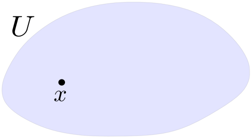


$$T \cong U \subset \mathfrak{t}^*$$

$$(\Sigma(M)_T, \Omega) \cong (\mathbb{T}^d \ltimes \mathfrak{t}^*, d\lambda)_U$$

- Lagrangian fibration with compact connected fibers $\Rightarrow (T, \Lambda_T)$ i.a. chart;
- $\Lambda_T = \ker(\exp) \subset \mathfrak{t}$.

- Integrability of (M, π) regular [Crainic-Fernandes]:



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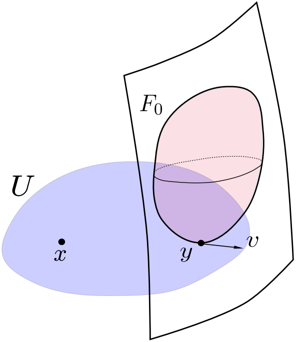

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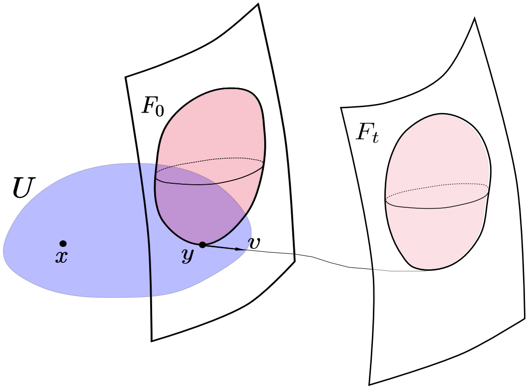
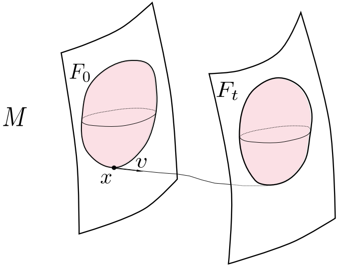


$$\exists\, U,\ \delta > 0, \forall\, v,\ |v| = 1 \quad A(t) = \int_{F_t} \omega_t \;\Rightarrow\; |A'(0)| \notin (0, \delta)$$

▷ Infinitesimal approach to (B,Λ) ($\Lambda_x \subset \mathfrak{t} = \ker(\exp)$):



$$\partial\colon \pi_2(F_0,x) \to \nu^*(F_0) \cong \mathfrak{t}$$

▷ [Crainic-Fernandes]:

Monodromy group at x $:= \mathrm{Im}\partial$.

- (M,π) integrable iff monodromy groups are uniformly discrete.
- If so monodromy group at x $= \ker(\exp) \subset \mathfrak{t}$.

▷ I.a. structure from infinitesimal data:

Theorem

(M,π) regular is $\underline{s}$-proper iff

1. *leaves of $\mathcal{F}$ compact and 1-connected (more generally with finite fundamental group),*
2. *Λ_x is a full rank lattice for all $x \in M$.*

Moreover $(M,\mathcal{F},\Lambda_M)$ i.a. foliation, where

$$\Lambda_M := \bigcup_{x\in M} \Lambda_x$$

Of course Λ_M, induces the i.a. structure (B,Λ).

▷ Symplectic developing/period map for (M,π) PMCT:

$$\begin{tikzcd}
(u^*M,\pi^u) \arrow[r] \arrow[d] & (M,\pi) \arrow[d] \\
(\tilde{B},\tilde{\Lambda}) \arrow[r, "u"] & (B,\Lambda)
\end{tikzcd}
\qquad
\begin{tikzcd}
H^2(F_b;\mathbb{R}) \arrow[r, hook] & \mathcal{H}^2 \arrow[d, shift right] \\
 & (\tilde{B},\tilde{\Lambda}) \arrow[u, shift right, "{[\pi^u]}"']
\end{tikzcd}$$

$$D := \mathrm{pr}\circ[\pi^u]\colon (\tilde{B},\tilde{\Lambda}) \to (H^2(F;\mathbb{R}), H^2(F;\mathbb{Z})).$$ fig

$$\begin{tikzcd}
\pi_1(B,b) \arrow[rr, "\rho"] \arrow[dr] & & \mathrm{Aut}(H^2(F;\mathbb{Z})) \\
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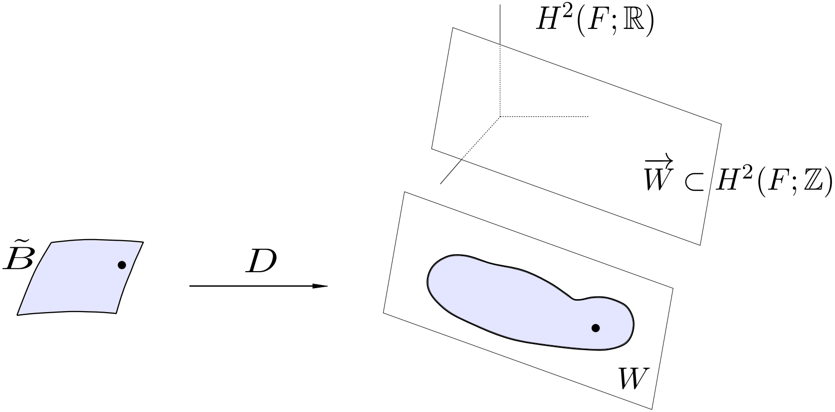

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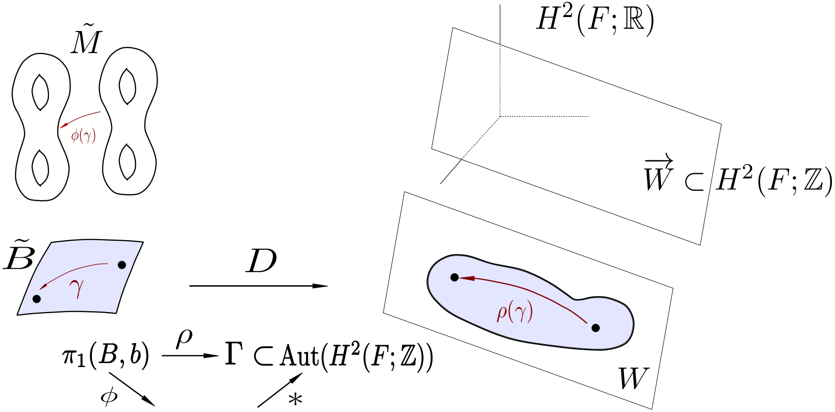
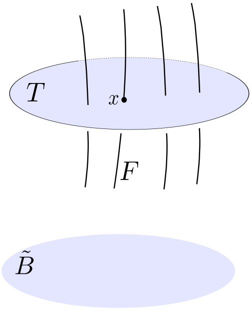


$$\begin{tikzcd} \pi_1(B,b) \arrow[rr, "\rho"] \arrow[dr, "\phi"'] & & \Gamma \subset \mathrm{Aut}(H^2(F;\mathbb{Z})) \\ & \mathrm{Diff}\,F/\mathrm{Diff}\,F^0 \arrow[ur, "*"'] & \end{tikzcd}$$

▷ The volume function $\mathbb{V}\colon H^2(F;\mathbb{R}) \to \mathbb{R}$ is constant on W.



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- The volume function is a polynomial [Duistermaat-Heckman];
- Compact i.a. manifolds can only support constant polynomials [Goldman-Hirsch].

▷ Uses of the symplectic developing map:

- Questions related to $H^2_\pi(M)$.
- Construction of PMCT: obstructions for F to be fiber of PMCT on $(H^2(F;\mathbb{R}), H^2(F;\mathbb{Z}))$ coming from existence of symplectic developing map.

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(M,π) regular $\underline{s}$-proper PM, leaves of $\mathcal{F}$ 1-connected.

▷ Back to $H^2_\pi(M)$:

- $H^2_\pi(M) \cong \Gamma(E)^{\Sigma(M)}, \quad E_x = H^2(\underline{s}^{-1}(x);\mathbb{R})$.

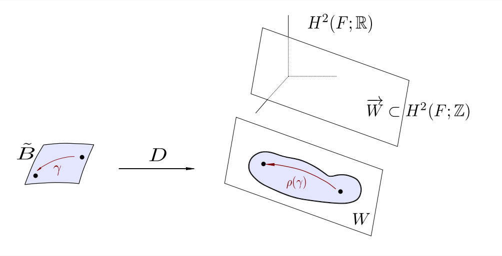
$$\begin{tikzcd} \mathbb{T}^d_x \arrow[r, hook] & \underline{s}^{-1}(x) \arrow[d] \\ & F_x \end{tikzcd} \qquad \pi_2(\underline{s}^{-1}(x)) \hookrightarrow \pi_2(F_x) \xrightarrow{\partial} \mathbb{Z}^d$$

$\pi_2(\underline{s}^{-1}(x)) \equiv$ Spheres in F_x with trivial transverse area variation.

$$\begin{tikzcd} H^2(F_b;\mathbb{R}) \arrow[r, hook] & \mathcal{H}^2 \arrow[r, two heads] \arrow[d] & \mathcal{H}^2/\overrightarrow{W} \arrow[d] \\ & (\tilde{B},\tilde{\Lambda}) \arrow[r] & (\tilde{B},\tilde{\Lambda}) \end{tikzcd}$$

- $H^2_\pi(M) \cong \Gamma(E)^{\Sigma(M)} = \Gamma(\mathcal{H}^2/\overrightarrow{W})^{\pi_1(B,b)}$.

- $[\pi] \neq 0 \in H^2_\pi(M)$ for PMCT revisited:



$$[\pi] \in H^2_\pi(M) \equiv w_b \in \Gamma(\mathcal{H}^2/\overrightarrow{W})^{\pi_1(B,b)},\ w_b \neq 0 \in H^2(F;\mathbb{R}).$$

Corollary

Let (M,π) be a s-proper PM with 1-connected symplectic leaves. Then if (B,Λ) is complete (more generally has non-trivial radiance obstruction class) then $[\pi] \neq 0 \in H^2_\pi(M)$.

▷ Poincaré duality pairing revisited:

$$H^k_\pi(M) \times H^{n-k}_\pi(M) \to E^{k\,\Sigma(M)} \times E^{n-k\,\Sigma(M)} \stackrel{\int_{s^{-1}(x)}}{\longrightarrow} C^\infty(M/\mathcal{F})$$

- Define

$$\mu_\Lambda := d\lambda_1 \wedge \cdots \wedge d\lambda_d \text{ (locally)}, \in \Gamma(\wedge^d T^* B).$$

Proposition

(M, π) oriented PMCT with 1-connected symplectic leaves. Then $H^*_\pi(M)$ supports a canonical non-degenerate Poincaré duality pairing

$$H^k_\pi(M) \times H^{n-k}_\pi(M) \to \mathbb{R}.$$

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- Regular Poisson structures near (M,π) PMCT (C^∞-topology):
  - $U_\pi \subset \mathrm{RegPoiss}(M)/\mathrm{Diff}^0(M) \to V_\pi \subset \mathrm{Poiss}(M,\mathcal{F})/\mathrm{Diff}^0(M,\mathcal{F})$.

    Compact foliations with fibers having finite π_1 are stable [Epstein].
  - $V_\pi \subset \mathrm{Poiss}(M,\mathcal{F})/\mathrm{Diff}^0(M,\mathcal{F}) \to \mathcal{V}_{[\pi]} \subset \Gamma(\mathcal{H})^{\pi_1(b,B)}/\mathrm{Diff}^0 B$.

    [Moser]
  - $\mathcal{U}_0 \subset H^2_\pi(M) \cong \Gamma(\mathcal{H}/\overrightarrow{W})^{\pi_1(B,b)} \to U_\pi \subset \mathrm{RegPoiss}(M)/\mathrm{Diff}^0(M)$.
  - Regular integrable Poisson structures near π are of compact type (monodromy lattice vary smoothly).

    They can be recognized as those π' for which $D([\tilde{\pi}']) \subset H^2(F;\mathbb{R})$ is contained in an affine subspace with integral affine directions.

    Apart from scaling, PMCT expected to be isolated (at least the i.a. structure Λ on B to be isolated).

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- $V_\pi \subset \mathrm{Poiss}(M,\mathcal{F})/\mathrm{Diff}^0(M,\mathcal{F}) \to \mathcal{V}_{[\pi]} \subset \Gamma(\mathcal{H})^{\pi_1(b,B)}/\mathrm{Diff}^0 B$.

  [Moser]
- $\mathcal{U}_0 \subset H^2_\pi(M) \cong \Gamma(\mathcal{H}/\overrightarrow{W})^{\pi_1(B,b)} \to U_\pi \subset \mathrm{RegPoiss}(M)/\mathrm{Diff}^0(M)$.
- Regular integrable Poisson structures near π are of compact type (monodromy lattice vary smoothly).

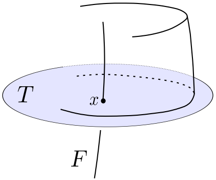
  They can be recognized as those π' for which $D([\tilde{\pi}']) \subset H^2(F;\mathbb{R})$ is contained in an affine subspace with integral affine directions.

  Apart from scaling, PMCT expected to be isolated (at least the i.a. structure Λ on B to be isolated).

- Assumption on leaves of $\mathcal{F}$ being 1-connected can be removed:
  - (B, Λ) carries i.a. orbifold structure.
  - The i.a. orbifold structure on (B, Λ) is a global quotient:
  - We do not know whether the volume polynomial has to be constant (i.e. we do not have an extension to the i.a. orbifold case of Goldman-Hirsch non-existence of non-trivial polynomials on compact i.a. manifolds).
- Dirac case:
  - Some of the previous discussion extends, including the developing map construction. The volume function is constant, but can be zero.

(M,π) non-regular $\underline{s}$-proper Poisson manifold.

▷ B is an i.a. manifold with polyhedral boundary [Zung]:



$$T \cong U \subset \mathfrak{g}^*$$

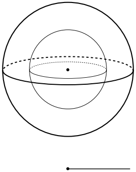
$$(\Sigma(M)_T, \Omega) \cong (G \ltimes \mathfrak{g}^*, d\lambda)_U$$

$$(B,\Lambda) \qquad (\mathfrak{g}^*/G \cong \mathfrak{t}^*/\mathcal{W}, \Lambda_{\mathrm{kerexp}}) \cong (\mathcal{W}_{\mathfrak{c}}, \Lambda_{\mathrm{kerexp}})/(G/G^0)$$

▷ $V\colon B^{\mathrm{pri}} \to \mathbb{R}^+$ polynomial converging to zero in ∂B (semi-local models).

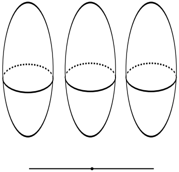
- Local regular resolution of (B, Λ) (Weyl's covering theorem):

$(\mathfrak{g}^*, \pi_{\mathrm{lin}})$



$\mathfrak{t}^*/\mathcal{W}$

$(G/T \times \mathfrak{t}^*, L_{\mathrm{lin}})$

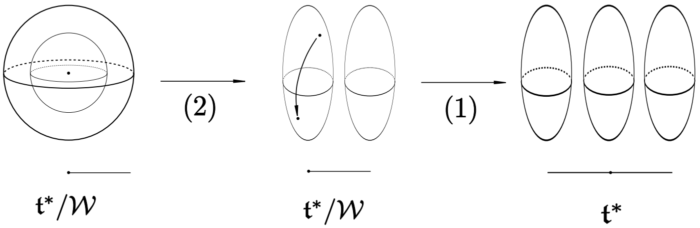


$\mathfrak{t}^*$

L_{lin};-$\langle \xi, [u+\mathfrak{t}, v+\mathfrak{t}] \rangle_{([T],\xi)}$, *and* G$-$*action*

- Local regular resolution of (B,Λ) (Weyl's covering theorem):

$$(\mathfrak{g}^*,\pi_{\mathrm{lin}}) \longleftarrow (G/T\times\mathfrak{t}^*,L_{\mathrm{lin}})/\mathcal{W} \longleftarrow (G/T\times\mathfrak{t}^*,L_{\mathrm{lin}})$$



L_{lin};-$\langle\xi,[u+\mathfrak{t},v+\mathfrak{t}]\rangle_{([T],\xi)}$, *and* G$-$*action*

(1) From M to M^{hol}.

(2) Over each $x\in M$ collect the set of maximal tori in G_x.

▷ **Regular resolution** of $\underline{s}$-proper symplectic groupoid $(\mathcal{G},\Omega) \rightrightarrows (M,\pi)$:

$$\mathcal{G}^{\mathrm{r}} = \{(g, T) \,|\, g \in \mathcal{G},\ T < G_x \,\text{maximal torus},\ x = \underline{s}(g)\}.$$

- The units are

$$Y = \{(x, T) \,|\, x \in M,\ T < G_x \,\text{maximal torus}\}.$$

- $\mathcal{G}^{\mathrm{r}}$ has an obvious groupoid structure and

$$\begin{array}{rcl} r\colon \mathcal{G}^{\mathrm{r}} & \longrightarrow & \mathcal{G} \\ (g, T) & \longmapsto & g \end{array}$$

is a surjective morphism of groupoids, which restricts to a bijection

$$r^{-1}(\mathcal{G}^{\mathrm{reg}}) \to \mathcal{G}^{\mathrm{reg}}.$$

sk

▷ The smooth structure on $\mathcal{G}^{\mathrm{r}}$:

- A groupoid isomorphism $\phi\colon \mathcal{G} \to \mathcal{G}'$ induces

$$\begin{tikzcd} \mathcal{G}^{\mathrm{r}} \arrow[r, "\Phi"] \arrow[d, "r"] & \mathcal{G}^{\mathrm{r}\prime} \arrow[d, "r'"] \\ \mathcal{G} \arrow[r, "\phi"] & \mathcal{G}' \end{tikzcd}$$

- Use "linear charts" for $\mathcal{G}$. Show "change of coordinates" induces diffeomorphism.

$$\begin{tikzcd} G/T \times \mathfrak{t}^* \times T \times G/T \times U \arrow[rr, "\Phi_1?"] \arrow[d, "r"] & & G'/T' \times \mathfrak{t}^* \times G' \times U' \arrow[d] \\ \mathfrak{g}^* \times G \times U \arrow[r, "\phi"] & \mathfrak{g}'^* \times G' \times U' \arrow[r, "\pi_2"] & G' \end{tikzcd}$$

- G/T (resp. G'/T') parametrizes adjoint family of maximal tori.
- Φ_1 parametrizes a (smooth) family of maximal tori in G'; any family of maximal tori is obtained by (smooth) pullback of the adjoint family
- Enough $H^1_{\mathrm{diff}}(T', \mathfrak{g}'/\mathfrak{t}') = 0$ (T' compact) [Coppersmith].

Theorem

$(\mathcal{G},\Omega) \rightrightarrows (M,\pi)$ symplectic $\underline{s}$-proper groupoid. The regular resolution $(\mathcal{G}^{\mathrm{r}}, r^*\Omega) \rightrightarrows (Y, L)$ is a $\underline{s}$-proper presymplectic regular groupoid. In the commutative diagram

$$\begin{tikzcd} (\mathcal{G}^{\mathrm{r}}, r^*\Omega) \arrow[r, "r"] \arrow[d, shift left] \arrow[d, shift right] & (\mathcal{G},\Omega) \arrow[d, shift left] \arrow[d, shift right] \\ (M^r, L^r) \arrow[r, "r"] & (M,\pi) \end{tikzcd}$$

- *the top horizontal arrow is a surjective proper Lie groupoid morphism and the horizontal arrows are backward Dirac;*
- *$r\colon (M, L^r) \to (M,\pi)$ induces a homeomorphism of leaf spaces which is an integral affine diffeomorphism over the regular subset.*

Corollary

(M,π) $\underline{s}$-proper. Then its leaf space admits a structure of i.a. orbifold (upgrading Zung's i.a. structure with polyhedral boundary).

Theorem

A PMCT (M, π) must be regular.

- Resolve (M, π) to (M^r, L^r), the latter having leaf space B with i.a. orbifold structure.
- $V\colon M^{\mathrm{pri}}/\mathcal{F} \cong B^{\mathrm{pri}} \to \mathbb{R}^+$ induces non-constant polynomials $\mathbb{V}\colon B^{\mathrm{hol}} \to \mathbb{R}$ and $\mathbb{V}_D\colon \mathbb{R}^d \to \mathbb{R}$, the latter Γ-equivariant (up to sign) and with zero set containing integral affine hyperplanes (the fixed points of the reflections on simple roots).

- Dirac case:
  - There is a regular resolution of (M, L) $\underline{s}$-proper Dirac manifold.
  - Its leaf space carries i. a. orbifold structure.
  - Non-existence for Dirac manifolds of CT with a non-constant polynomial constructed out of the leafwise presymplectic form and global closed forms on M (for example adding a global closed 2-form).

Proposition

Let $\mu\colon (X,\sigma) \to \mathbb{T}^d$ be a free quasi-Hamiltonian Abelian space. Then the reduced Poisson space (M, π_{red}) is of CT iff $\pi_1(X) \cong \mathbb{Z}^d$. Its leaf space is the standard i.a. torus $(\mathbb{R}^d/\mathbb{Z}^d, \mathbb{Z}^d)$.

▷ There exists a free quasi-Hamiltonian $\mathbb{T}^1$-space. The symplectic fiber of its Poisson reduced space is diffeomorphic to the K3 surface [Kotschick].

Theorem

There exists PMCT whose symplectic fiber is diffeomorphic to the K3 surface. The leaf space is $(\mathbb{R}^2/\Theta, \mathbb{Z}^2)$ where up to scaling Θ is any lattice of full rank in $\mathbb{Q}^2$.

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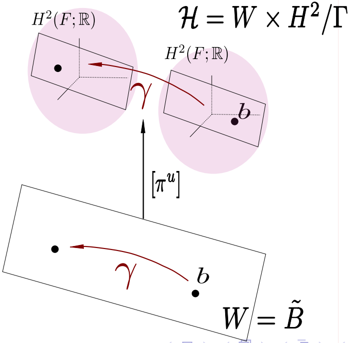
- Construction of (complete) PMCT with fiber F in 3 steps:
  1. Find $W \subset H^2(F;\mathbb{R})$, $\Gamma \subset \mathrm{Aut}(H^2(F;\mathbb{Z}))$ so that (i) $\overrightarrow{W} \subset H^2(F;\mathbb{Z})$, (ii) $\mathrm{Vol}_{|W} = \mathrm{constant} \neq 0$ (iii) $\Gamma(W) \subset W$ and the induced action is free, properly discontinuous and co-compact.



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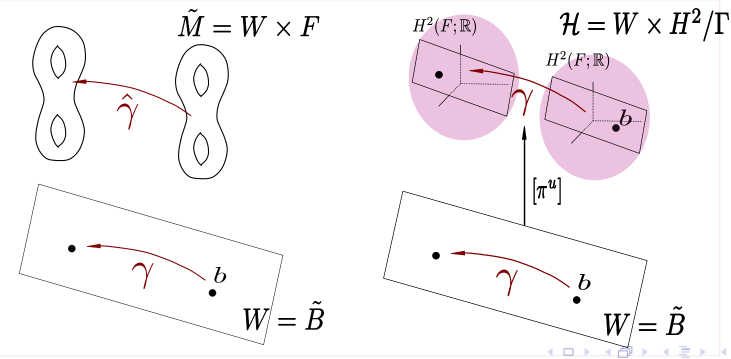
  2. Find $\hat{\Gamma}$ image of a right inverse to $\mathrm{Diff}(F) \to \mathrm{Aut}(H^2(F;\mathbb{Z}))$ over Γ.



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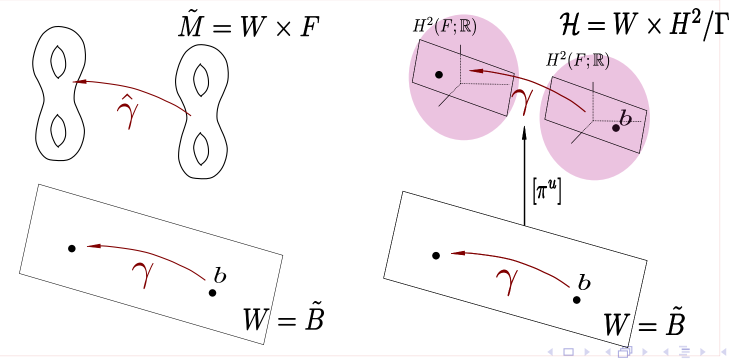
  3. Find $\hat{W}$ image of a Γ-$\hat{\Gamma}$-equivariant right inverse to $\Omega^2_{\mathrm{symp}}(F) \to H^2(F;\mathbb{R})$ over W (not needed for Dirac manifolds).



▷ One solution to the cohomological problem:

Assume F 1-connected has dimension 4, $\mathbb{L} := (H^2(F;\mathbb{Z}), \cup)$ and $\mathrm{Aut}(\mathbb{L})$ are the automorphisms of the cohomology ring.

- $\mathrm{Vol}_{|W} = \mathrm{constant}$ rules out many intersection forms.
- Let H be the hyperbolic intersection form (the intersection form of $S^2 \times S^2$); in the basis x, y its matrix is
$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$
- $W(k)^2 \subset 3H$, $k > 0$, $\Gamma \subset \mathrm{Aut}(3H)$, $\overrightarrow{W} \subset H^2(F;\mathbb{Z})$, $\Gamma_{|W}$ free, properly discontinuous and co-compact (all posibilities on linear holonomy).
- Same with $W(k)^2 \subset 3H \oplus \mathbb{L}'$

▷ Nielsen realization problem:

F^4 1-connected with intersection form $\mathbb{L}$. Study lifts over subgroups for

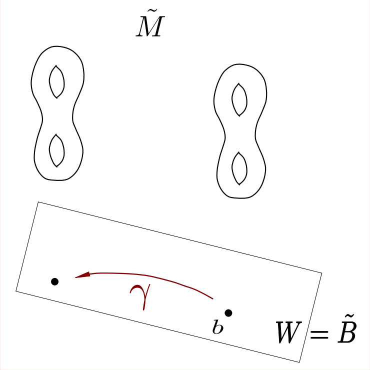
$$\mathrm{Diff}(F) \to \pi_0(\mathrm{Diff}(F)) \to \mathrm{Aut}(\mathbb{L}).$$

- If $\Gamma = \mathbb{Z} \cdot \gamma$, and $F = F' \# S^2 \times S^2$, F' with indefinite intersection form, lifts always exist [Wall].

  If F' has intersection for $H \oplus \mathbb{L}'$, then F has intersection form $2H \oplus \mathbb{L}'$ and we can construct (M, L) Dirac manifold of compact type with leaf space $(R/k\mathbb{Z}, \mathbb{Z})$.

- If Γ has relations other geometric structures needed to bring rigidity.

Let $F = K3$, $H^2(F;\mathbb{Z}) \cong 3H \oplus -2E_8$:



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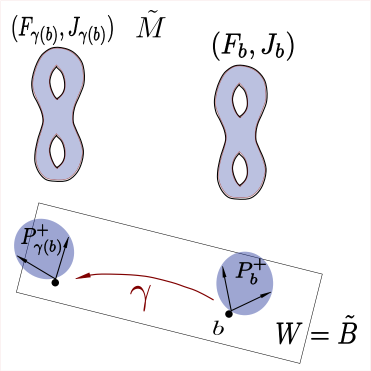

1. P^+_b positive oriented plane, Γ-equivariant.

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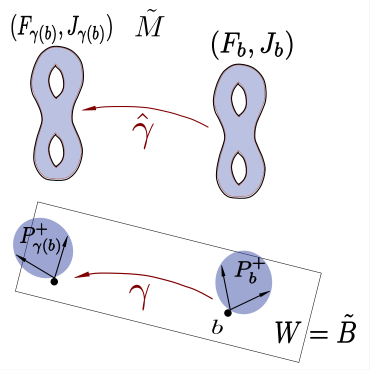
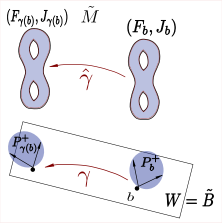


1. P^+_b positive oriented plane, Γ-equivariant.
2. If b Kahler class for (F_b, J_b), there is a unique lift of γ

$$\hat{\gamma}\colon (F_b, J_b) \to (F_{\gamma(b)}, J_{\gamma(b)})$$

A necessary condition is $\mathrm{span}\langle p, P^+_b\rangle \in \mathrm{Gr}^+_3$.



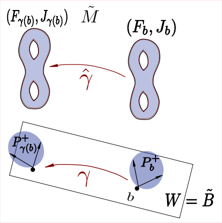
- Only get Γ-equivariant family $\mathrm{span}\langle p, P_b^+\rangle \in \mathrm{Gr}_3^+$

$$\Gamma = \langle \begin{pmatrix} 0 \\ n \end{pmatrix} + \mathrm{I}, \begin{pmatrix} m \\ 0 \end{pmatrix} + \mathrm{I} \rangle, \; n, m \in \mathbb{Z}.$$

- Must have $\mathrm{span}\langle p, P_b\rangle \subset \mathrm{Gr}_3^+ \backslash \bigcup_{i\in\mathbb{N}} H_i$, H_i has codimension 3.

  Possible to find such families of 3-planes (explicitly computation).

3. $\mathrm{span}\langle p, P_b\rangle \in \mathrm{Gr}_3^+ \backslash \bigcup_{i\in\mathbb{N}} H_i$ defines a hyperkahler metric in (F_b, J_b). The harmonic representative ω_b of b is a symplectic form (a Kahler form), and this finishes the construction of the PMCT.



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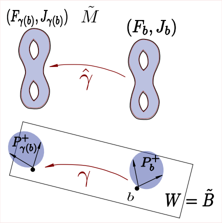

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▷ Final remarks:

1. We would like to realize more i.a. leaf spaces.
   - Modifications: Products, blow up a section, replace $K3$ by $K3^{[n]}$.
   - No orbifold leaf spaces so far. Our examples have finite groups of automorphisms (order 2 and 4), but with fixed points.

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2. PMCT and compact isotropic symplectic realizations (CISR):
   - A CISR (with connected fiber) $(X, \sigma) \to (M, \pi)$ induces Θ a transverse i.a. structure on (M, π) [Delzant-Dazord]
   - Free quasi-Hamiltonian $\mathbb{T}^d$-space $\cong$ CISR with trivial linear holonomy.
   - (M, π) compact with characteristic foliation a fibration $(M, \pi) \to B$, [Delzant-Dazord] looked at the existence of CISR realizing Θ a given transverse i.a. structure.
   - If the fiber F of $(M, \pi) \to B$ has finite fundamental group, then

$$\Lambda < \Theta,$$

     where Λ is the monodromy lattice.
     For a PMCT Λ the monodromy lattice coarsest possible transverse i.a. structure coming from a CISC.
   - If (M, π) PMCT, and (X, σ) CISR, then

$$(\Sigma(M), \Omega) \cong (X, \sigma) \times (X, -\sigma)/\mathcal{I}$$

     comes from "finite dimensional Hamiltonian reduction".

Theorem

There is a 1 to 1 correspondence between

- *Free quasi-Hamiltonian $\mathbb{T}^d$-space with fundamental group $\mathbb{Z}^d$*
- *PMCT with leaf space $(\mathbb{T}^d, \mathbb{Z}^d)$.*

▷ *Modification of PMCT with leaf space $(\mathbb{T}^d, \mathbb{Z}^d)$: fuse with (Z, ϖ) 1-connected $\mathbb{T}^d$-Hamiltonian space (Toric variety).*

▷ *More generally a PMCT with leaf space $(\mathbb{R}^d/\Theta, \mathbb{Z}^d)$, $\Theta < \mathbb{Q}^d$ admits CISR.*

▷ Split the problem into

- $\Sigma(M) \rightrightarrows M$ being "elementary" (classification of regular groupoids [Moerdijk]).
- Deal with the multiplicative symplectic form question.

3. Features of the regular resolution:
    - Valid for proper presymplectic groupoids [Crainic-Struchiner, Pflaum-Posthuma-Tang].
    - Generalizes to any proper groupoid to a partial regular resolution: replace maximal torus by the connected component of the isotropy group of a regular point (its adjoint orbit is well defined inside any isotropy subgroup).
    - Partial resolution is minimal among regular resolutions.
    - It is invariant of the Morita equivalence class.
    - It is an equivariant (partial) resolution of singularities in the sense of Laurent-Gengoux ($\mathcal{G}$ acts in the resolution).

4. Poisson manifolds with compact symplectic integrations (honest compact type):

Theorem

Let $(\mathcal{G}, \Omega)$ be a compact symplectic groupoid. Then it must be regular.

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Let $(\mathcal{G}, \Omega)$ be a compact presymplectic groupoid. Then if it supports a non-zero volume polynomial it must be regular.

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- For G a compact connected Lie group, we have the AMM twisted presymplectic groupoid

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- Lagrangian fibration $(X, \sigma) \to B$ with compact connected fibers having a global Lagrangian section is a symplectic groupoid integrating $(B, \pi = 0)$.

  A $\underline{s}$-proper ($\underline{s}$-connected) symplectic groupoid $(\mathcal{G}, \Omega)$ integrating $(B, \pi = 0)$ induces an i.a. structure on its leaf space B [Duistermaat].

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A $\underline{s}$-proper (twisted pre)symplectic groupoid $(\mathcal{G}, \Omega)$ induces an orbifold i.a. structure on its leaf space B.

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