Friday fish: reading seminar on four-manifolds 1

Introduction

Ralph Klaasse, January 24, 2014

This talk is the first in a reading seminar during Friday Fish, set-up to study the book by Gompf and Stipsicz - 4-manifolds and Kirby calculus [4], roughly from front to finish. In this talk I will introduce the topic and discuss the aim of the seminar. I will then discuss some results, mainly before the introduction of Seiberg-Witten gauge theory into the subject. The contents of this latter portion of the talk can mostly be found in Chapter 1 of [4].

1 Overview

The structure of this talk is as follows.

- Introduction: aim of the seminar;
- The intersection form;
- Freedman's and Donaldson's theorems and consequences;
- Closing: characteristic classes;
- Distribution of upcoming talks.

2 Introduction: aim of the seminar

As is mentioned in the outline of this seminar, the aim is to understand the study of four-manifolds mainly through a technique called Kirby calculus. Essentially, one decomposes a given four-manifold into balls and studies the attaching maps. The resulting handlebody decomposition can then be viewed as a diagram, a Kirby diagram, and the Kirby calculus (1970s) consists of "Kirby moves" on such diagrams with which one can prove two four-manifolds are isomorphic. The study of handlebodies is equivalent to Morse theory. However, this will have to wait until Part II of the book, i.e. Chapter 4.

Let me first mention my personal interest in the book. As was mentioned in the outline, much progress in the study of four-manifolds was made using Donaldson and Seiberg-Witten gauge theory. Indeed, the following is a direct quote of the first sentences from the preface of [4]:

The past two decades represent a period of explosive growth in 4-manifold theory. From a desert of nearly complete ignorance, the theory has flourished into a virtual rain forest of ideas and techniques, a lush ecosystem supporting complex interactions between diverse fields such as gauge theory, algebraic geometry and symplectic topology, in addition to more topological ideas.

Very roughly, as the more classical algebro-topology invariants such as the homology and cohomology groups of the four-manifold are in fact homotopy invariants of the manifold and hence cannot be used to distinguish the diffeomorphism type, one instead considers the space of solutions to a PDE on sections of a vector bundle on the manifold, and studies its (co)homology pairing. Donaldson theory does this on an SU(2)-bundle $E \to X$ with PDE given by the anti-self-duality equations $F_A^+ = 0$ for $A \in \mathrm{Conn}(E)$ a unitary connection, while Seiberg-Witten theory instead uses a $\mathrm{Spin}^{\mathrm{c}}$-structure to create two U(2)-bundles S^+ and S^- called spinor bundles, and then studies solutions (A, ψ) to the equations $F_A^+ = q(\psi, \psi)$, $D_A\psi = 0$, where $A \in \mathrm{Conn}(\bigwedge^2 S^+)$ and $\psi \in \Gamma(S^+)$, and D_A is the Dirac operator on S^+.

Donaldson theory was started around 1983 when he proved what is now called Donaldson's theorem, while Seiberg-Witten theory began in 1994. Soon after Witten wrote his famous paper, experts on Donaldson theory realized the resemblance and used the new equations to quickly reprove results attained by Donaldson theory and more. In this sense, for most of the applications covered in this book, Donaldson theory has been superseded by Seiberg-Witten theory. You will see quite some mention of results from Seiberg-Witten theory in the book, which even includes an (albeit very brief) introduction. However, gauge theory is not our main concern in this seminar. If there is interest I can provide references for an introduction to either gauge theory, and perhaps later on it might be nice to give a more modern proof of Donaldson's theorem using Seiberg-Witten theory.

Nevertheless, the gauge theoretic results are mostly negative, in that they show that certain combinations of geometric structures are incompatible, or indeed that certain four-manifolds are *not* diffeomorphic. In contrast, Kirby calculus provides a way of arriving at positive existence results, by quite explicit constructions. Kirby calculus was around before gauge theory, but got revived after gauge theory showed what was not attainable, thus guiding the way by showing what route *not* to take.

I should also perhaps mention the following book,

- A. Scorpan - The wild world of 4-manifolds (2005),

which describes itself not as a textbook but as a travel guide through the techniques and flavor of the theory on four-manifolds. Its author was a graduate student of Kirby at Berkeley along with David.

3 The intersection form

From now on we restrict ourselves to compact orientable four-manifolds X without boundary, in principle just topological. First off, let us note that classifying such X even up to homeomorphism can never be done, as the following theorem exists, along with the negative solution of the word problem for groups[1].

[1]This is the Novikov-Boone theorem, which states that given a finite presentation of a group there is no algorithmic procedure to decide when two words describe the same element.

Theorem 3.1. *Any finitely-presented group is the fundamental group of a compact (even symplectic) four-manifold.*

Hence we will sometimes assume X to be simply-connected, but we will explicitly state this. Let us first describe the integral (co)homology of X. We know that $H_0(X;\mathbb{Z}) \cong H_4(X;\mathbb{Z}) \cong \mathbb{Z}$. If we assume X simply-connected, $\pi_1(X) = 0$, then by Hurewicz' theorem we get after abenializing also $H_1(X;\mathbb{Z}) = 0$. But then from Poincaré duality we get $H_3(X;\mathbb{Z}) = 0$, so that all that remains is $H_2(X;\mathbb{Z})$. But now note that by the universal coefficient theorem $H^2(X;\mathbb{Z}) \cong \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z})$, so $H^2(X;\mathbb{Z}) = \mathbb{Z}^{b_2(X)}$, whence by Poincaré duality we also have $H_2(X;\mathbb{Z}) = \mathbb{Z}^{b_2(X)}$.[2] Let us now define what is called the intersection form, which essentially uses that for $n = 4$ we have $2 + 2 = 4$. Recall that X be compact oriented means it has a fundamental class $[X] \in H_4(X;\mathbb{Z})$.

Definition 3.2. [4, Definition 1.2.1] Let X be a compact oriented topological four-manifold. The symmetric bilinear form $Q_X : H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to \mathbb{Z}$, defined by

$$Q_X(a,b) = \langle a \cup b, [X]\rangle = a \cdot b \in \mathbb{Z}, \qquad a,b \in H^2(X;\mathbb{Z}), \tag{3.2}$$

is called the *intersection form* of X.

Note that through Poincaré duality we could have defined it on $H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z})$ or $H^2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z})$ as well. Note that Q_X is defined using purely topological information.

Remark 3.3. Note that by bilinearity we have $Q_X(a,b) = 0$ if either a or b is torsion. Indeed, if $na = 0$ for some $n \in \mathbb{N}$ then by bilinearity we have

$$0 = Q_X(0,b) = Q_X(na,b) = nQ_X(a,b). \tag{3.3}$$

Hence Q_X descends to $H_2(X;\mathbb{Z})/\mathrm{Torsion}$. But then we can represent Q_X by a matrix; we denote the determinant of this matrix by $\det Q_X$.

Let us be a bit more explicit about the definition of Q_X in the smooth case.

Theorem 3.4. *[4, Proposition 1.2.3] Let X be a compact oriented smooth four-manifold. Then every element $\alpha \in H_2(X;\mathbb{Z})$ can be represented by an embedded surface, i.e. there exists a compact oriented surface Σ and an embedding $i : \Sigma \hookrightarrow X$ such that $i_*([\Sigma]) = \alpha$, where $[\Sigma] \in H_2(\Sigma;\mathbb{Z})$ is the fundamental class of Σ.*

Proof. Let $\alpha \in H_2(X;\mathbb{Z})$ be given, and let $\mathrm{PD}(\alpha) \in H^2(X;\mathbb{Z})$ be its Poincaré dual. Then by the classification of U(1)-bundles over X by their first Chern class, elements of $H^2(X;\mathbb{Z})$ are in fact in one-to-one correspondence with such U(1)-bundles over X. Let $L_\alpha \to X$ be the one corresponding to $\mathrm{PD}(\alpha)$ and consider a generic section $s \in \Gamma(L_\alpha)$. Then the zero set $Z_\alpha = \{x \in X : s(x) = 0\}$ will be a smooth surface with $[Z_\alpha] = \alpha$. $\square$

[2] In general, we have $H_i(X;\mathbb{Z}) \cong \mathbb{Z}^{b_i(X)} \oplus T_i$ where T_i is the torsion, and then

$$\begin{aligned} \mathrm{Hom}(H_i(X;\mathbb{Z}),\mathbb{Z}) &\cong \mathrm{Hom}(\mathbb{Z}^{b_i(X)},\mathbb{Z}) \oplus \mathrm{Hom}(T_i,\mathbb{Z}) \cong \mathbb{Z}^{b_i(X)}, \\ \mathrm{Ext}(H_i(X;\mathbb{Z}),\mathbb{Z}) &\cong \mathrm{Ext}(\mathbb{Z}^{b_i(X)},\mathbb{Z}) \oplus \mathrm{Ext}(T_i,\mathbb{Z}) \cong T_i. \end{aligned} \tag{3.1}$$

The universal coefficient theorem then implies that $H^i(X;\mathbb{Z}) \cong Z^{b_i(X)} \oplus T_{i-1}$, but for $i = 2$ we know T_1 vanishes by simply-connectedness.

We see that there is some freedom here, and in fact this will be studied using the *genus function* by Joey when he covers Chapter 2.

Remark 3.5. Note that if X is simply-connected, then by the Hurewicz theorem we get $\pi_2(X) \cong H_2(X;\mathbb{Z})$, so that for compact simply-connected four-manifolds all elements $\alpha \in H_2(X;\mathbb{Z})$ can be represented by *immersed spheres*. These are in general not embeddings, but one can assume that $S^2 \to X^4$ has only transverse double points of self-intersection.

So now given $a, b \in H^2(X;\mathbb{Z})$, Poincaré duals $\alpha = \mathrm{PD}(a)$, $\beta = \mathrm{PD}(b) \in H_2(X;\mathbb{Z})$ and surface representatives Σ_α and Σ_β respectively, note that both these surfaces inherit orientations from their respective abstract surfaces.

Definition 3.6. Given generic $\Sigma_\alpha, \Sigma_\beta$, i.e. all their intersections are transverse, for each point $x \in \Sigma_\alpha \cap \Sigma_\beta$, consider the decomposition of tangent spaces

$$T_x\Sigma_\alpha \oplus T_x\Sigma_\beta = T_xX. \tag{3.4}$$

All three terms above now have orientations, and we can by concatenating bases compare orientations on both sides. We define $\varepsilon(x) = \pm 1$ depending on whether these orientations agree or disagree. The *intersection* of $\Sigma_\alpha, \Sigma_\beta$ is then defined to be

$$\Sigma_\alpha \cdot \Sigma_\beta = \sum_{x \in \Sigma_\alpha \cap \Sigma_\beta} \varepsilon(x) \in \mathbb{Z}. \tag{3.5}$$

Note that the definition of ε does not depend on the order of $\{\alpha, \beta\}$. We then have the following result explaining the name for Q_X.

Proposition 3.7. *[4, Proposition 1.2.5] Let X be a compact oriented smooth four-manifold. Given $a, b \in H^2(X;\mathbb{Z})$ and $\alpha, \beta, \Sigma_\alpha, \Sigma_\beta$ as above, with the latter two having transverse intersections, we have*

$$Q(a, b) = \Sigma_\alpha \cdot \Sigma_\beta. \tag{3.6}$$

Consider now the map $i : H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R})$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$. If we let $\omega, \eta \in \Omega^2(X;\mathbb{Z})$ be closed forms such that $[\omega] = i(a)$ and $[\eta] = i(b)$, we further have that

$$Q_X(a, b) = \int_X \omega \wedge \eta = \int_{\Sigma_\alpha} \eta = \int_{\Sigma_\beta} \omega. \tag{3.7}$$

The next important result easily proven using Poincaré duality is the following.

Theorem 3.8. *Let X be a compact oriented topological four-manifold. Then the intersection form Q_X is non-degenerate on $H_2(X;\mathbb{Z})/\mathrm{Torsion}$.*

It is time for some examples of intersection forms of well-known manifolds.

Example 3.9.

- S^4 has $b_2 = 0$, so that $Q = 0$;
- $\mathbb{C}P^2$ has a well-known cell-decomposition with one cell in dimension 0, 2 and 4, so that $H_2(\mathbb{C}P^2;\mathbb{Z}) = \mathbb{Z}$. We then get $Q = \langle 1 \rangle$, as all complex submanifolds intersect positively, and lines intersect at exactly one point.

- $\overline{\mathbb{C}P}^2$, the complex projective plane with reversed orientation, then has $Q = \langle -1 \rangle$;

- $S^2 \times S^2$, which has $H_2(S^2 \times S^2; \mathbb{Z})$, and we can generate it by the classes of $a = S^2 \times \{p\}$ and $b = \{q\} \times S^2$, where p, q are arbitrary points. Hence $a \cdot a = b \cdot b = 0$ as there are no intersection points after adjusting to $p \mapsto p'$, $q \mapsto q'$. Further, $a \cdot b = 1 = b \cdot a$ as there is just the point of intersection (q, p). We conclude that $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =: H$;

- $K3$, the Calabi-Yau manifold of lowest dimension, has $Q = 3H + 2E_8$, where E_8 is the matrix
$$E_8 = \begin{pmatrix} 2 & 0 & 1 & & & & & \\ 0 & 2 & 0 & 1 & & & & \\ 1 & 0 & 2 & 1 & & & & \\ & 1 & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & 1 \\ & & & & & & 1 & 2 \end{pmatrix}. \tag{3.8}$$
We see that $b_2(K3) = 22$.

- Given two compact oriented smooth manifolds X_1, X_2, one can consider their connected sum $X = X_1 \# X_2$, which has using Mayer-Vietoris
$$H_2(X;\mathbb{Z}) \cong H_2(X_1;\mathbb{Z}) \oplus H_2(X_2;\mathbb{Z}) \qquad \text{and} \qquad Q_X = \begin{pmatrix} Q_{X_1} & 0 \\ 0 & Q_{X_2} \end{pmatrix}, \tag{3.9}$$
after picking bases for X_i. This uses the fact that the cycles or surfaces representing the homology classes generically miss the point used for the connected sum. Generically, then, the intersection form will be diagonal as there is no intersection between cycles from different summands.

Recall that we can represent Q_X by a $b_2 \times b_2$-matrix, which will have coefficients in $\mathbb{Z}$. We now briefly discuss some notions from the theory of integral forms. Note that symmetric matrices with real entries are diagonalizable over $\mathbb{R}$.

Definition 3.10. Given a symmetric bilinear form Q on a finitely generated abelian group A, considers its extension over $A \otimes_{\mathbb{Z}} \mathbb{R}$. Let $b_2^{\pm}$ denote the number of ± 1's on the diagonal, and call the difference
$$\sigma(Q) = b_2^+ - b_2- \tag{3.10}$$
the *signature* of Q. Moreover, call the dimension of A the *rank* of Q, $\mathrm{rk}(Q)$. We call Q *positive/negative definite* if $\mathrm{rk}(Q) = \pm\sigma(Q)$, and *indefinite* otherwise. The form Q is called *unimodular* if $\det(Q) = \pm 1$, *even* if $Q(a, a) \equiv 0 \mod 2$ for all $a \in A$ and *odd* otherwise.

Corollary 3.11. *Let X be a compact oriented topological four-manifold. Then the intersection form Q_X on $H_2(X;\mathbb{Z})/\mathrm{Torsion}$ is unimodular, and $b_2(X) = b_2^+(Q_X) + b_2^-(Q_X)$. If one changes orientation of X, $b_2^{\pm}$ swap.*

We wish to study what type of unimodular forms can arise as the intersection form of a four-manifold. Let us first note the following classification theorem.

Theorem 3.12. *[4, Theorem 1.2.14] Let Q_1, Q_2 be two indefinite unimodular forms such that they have the same rank, signature and parity. Then Q_1 and Q_2 are equivalent.*

Now one may ask if any triple can be realized for indefinite unimodular forms. Note firstly that $|\sigma(Q)| < \mathrm{rk}(Q)$ and $\sigma(Q) \equiv \mathrm{rk}(Q) \mod 2$. But there are more restrictions.

Definition 3.13. Given a unimodular form Q on A, an element $a \in A$ is called *characteristic* if $Q(a,a) = Q(a,x) \mod 2$ for all $x \in A$.

Proposition 3.14. *[4, Lemma 1.2.20] Let Q be a unimodular form Q on A and $a \in A$ characteristic. Then $Q(a,a) = \sigma(Q) \mod 8$. In particular, if Q is even, $\sigma(Q)$ is divisible by 8.*

Proof. If a is characteristic for (A, Q), then $a + e_+ + e_-$ is characteristic for $(A \oplus \mathbb{Z} \oplus \mathbb{Z}, Q \oplus (1) \oplus (-1))$. But by Theorem 3.12, we then have

$$Q' = Q \oplus (1) \oplus (-1) \cong (b_2^+ + 1)\langle 1 \rangle \oplus (b_2^- + 1)\langle -1 \rangle, \tag{3.11}$$

and we note that any characteristic element for Q' has odd components. Now note that the square of any odd number is equal to 1 modulo 8. This means that

$$Q(a,a) = Q'(a + e_+ + e_-, a + e_+ + e_-) = (b_2^+ + 1) - (b_2^- + 1) = \sigma(Q) \mod 8. \tag{3.12}$$

If Q is even, 0 is characteristic, so that $\sigma(Q) \equiv 0 \mod 8$. □

Now note we have found all such indefinite intersection forms. Let Q be an indefinite unimodular form.

- If Q is odd, then we have

$$\begin{aligned} b_2^+ &= \frac{1}{2}\left(\mathrm{rk}(Q) + \sigma(Q)\right) =: m, \\ b_2^- &= \frac{1}{2}\left(\mathrm{rk}(Q) - \sigma(Q)\right) =: n, \\ Q &\cong m\langle 1 \rangle \oplus n\langle -1 \rangle, \\ Q &= Q_X \text{ for } X = m\mathbb{C}P^2 \# n\overline{\mathbb{C}P}^2. \end{aligned} \tag{3.13}$$

- If Q is even, we have

$$Q \cong \frac{\mathrm{rk}(Q) - |\sigma(Q)|}{2} H \oplus \frac{\sigma(Q)}{8} E_8. \tag{3.14}$$

Remark 3.15. One can show that $H \cong -H$ and $E_8 \oplus (-E_8) \cong 8H$.

Sadly there is no nice classification of definite unimodular forms, other than for any given rank there are only finitely many. This number may be very large: there are for example more than 10^{50} definite forms of rank 40.

4 Freedman's and Donaldson's theorems and consequences

Now let X be a compact oriented simply-connected four-manifold. We ask ourselves how much information about X is contained in its intersection form Q_X. Whitehead was able to prove the following.

Theorem 4.1. *[4, Theorem 1.2.25](Whitehead, Milnor, 1958) Let X_1, X_2 be compact oriented simply-connected four-manifolds. Then X_1 and X_2 are homotopy equivalent if and only if $Q_{X_1} \cong Q_{X_2}$.*

However, the following theorem by Freedman from 1981 greatly strengthens this theorem.

Theorem 4.2. *[4, Theorem 1.2.27](Freedman, 1981 [2]) For every unimodular symmetric bilinear form Q there exists a compact simply-connected topological four-manifold X such that $Q_X \cong Q$. If Q is even, X is unique up to homeomorphism. If Q is odd, there are exactly two homeomorphism types of four-manifolds with Q as their intersection form. At most one of these types carries a smooth structure.*

Corollary 4.3. *Compact simply-connected smooth four-manifolds are determined up to homeomorphism by their intersection form.*

Corollary 4.4. *If a topological four-manifold X is homotopy equivalent to S^4, then X is homeomorphic to S^4.*

This is a very nice result. Now we may ask which simply-connected topological manifolds carry smooth structures, and if a given intersection form Q is represented by a smooth manifold, how many non-diffeomorphic smooth manifolds are there with that same intersection form? One result in this direction, restricting the possible intersection forms of smooth manifolds was already known earlier.

Theorem 4.5. *[4, Theorem 1.2.29](Rohlin, 1952 [7]) Let X be a compact simply-connected smooth four-manifold. Then if Q_X is even, equivalently if X is* Spin*, we have $\sigma(X) \equiv 0 \mod 16$.*

In other words, $Q = E_8$ with $\sigma(E_8)$ cannot be represented by a smooth manifold. This was known before Freedman, but people did not know whether it was representable by a topological manifold. By Freedman's theorem it is, by some topological manifold X_{E_8} say. Note that by Rohlin's theorem, $X_{E_8} \# X_{E_8}$ might still admit smooth structures, as its signature is 16. The next big breakthrough after Freedman came in 1982 when Donaldson introduced instantons to the study of four-manifolds.

Theorem 4.6. *[4, Theorem 1.2.30](Donaldson, 1982 [1]) Let X be a compact simply-connected smooth four-manifold. If Q_X is positive/negative definite, then Q_X is equivalent to $n\langle \pm 1 \rangle$.*

This deals with what definite intersection forms arise from smooth manifolds. As mentioned I could give a proof of this theorem using Seiberg-Witten theory. For indefinite intersection forms arising from smooth manifolds, we know already that the coefficient of E_8 must be even. Before Seiberg-Witten theory the following was already known.

Theorem 4.7. *If the intersection form $2mE_8 \oplus nH$ is realized by a compact simply-connected smooth four-manifold, then if $m > 0$ we have $n \geq 3$.*

Remark 4.8. Note that for $m = 1$ we have already shown that $n = 3$ is attained, by the example of $X = K3$.

Note that Donaldson's theorem implies that $X_{E_8} \# X_{E_8}$ does not admit smooth structures. This last theorem has since been extended to the following result.

Theorem 4.9. *[4, Theorem 1.2.31](Furuta 10/8, 2001 [3]) let X be a compact simply-connected smooth four-manifold. If $Q_X \cong 2mE_8 \oplus nH$ (i.e. if Q is even / X is spin, then*

$$n \geq 2|m| + 1. \tag{4.1}$$

This is called the 10/8 theorem as it says that $b_2(X) \geq 10/8|\sigma(X)| + 1$. It is proven using Seiberg-Witten theory. There is a conjecture by Matsumoto called the 11/8-conjecture saying that this inequality should read $b_2(X) \geq 11/8|\sigma(X)|$ instead. To show some of the weirdness of four-manifolds, we note the following two results.

Theorem 4.10. *There are uncountably many pairwise non-diffeomorphic smooth four-manifolds all homeomorphic to $\mathbb{R}^4$.*

Theorem 4.11. *[4, Theorem 1.2.32] There are infinitely many pairwise non-diffeomorphic simply-connected smooth four-manifolds corresponding to the intersection forms*

$$2n(-E_8) \oplus (4n-1)H, \qquad n \geq 1 \qquad \textit{and} \qquad (2k-1)\langle 1 \rangle \oplus n\langle -1 \rangle, k \geq 2, n \geq 10k - 1. \tag{4.2}$$

Finally, there is the results of Fintushel-Stern giving uncountably many smooth structures by knot surgery.

5 Closing: characteristic classes

Chapter 1 of [4] also contains an appendix, section 1.4, which deals with characteristic classes. I suppose most of these results will be known. I wish to mention the following two results.

Theorem 5.1. *[4, Theorem 1.4.15] For a given four-manifold X and almost-complex structure J, we have*

$$c_2(X) = e(X) \in H^4(X;\mathbb{Z}), \qquad c_1(X) \equiv w_2(X) \mod 2, \qquad c_1^2(X,J) = 3\sigma(X) + 2\chi(X). \tag{5.1}$$

Conversely, if we are given an $h \in H^2(X;\mathbb{Z})$ with $h^2 = 3\sigma(X) + 2\chi(X)$ and $h \equiv w_2(X)$ mod 2, there is an almost-complex structure J on X with $h = c_1(X,J)$.

Corollary 5.2. *S^4, $(S^2 \times S^2)\#(S^2 \times S^2)$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$ do not admit almost-complex structures. A compact simply-connected four-manifold X admits an almost-complex structure if and only if $b_2^+(X)$ is odd.*

Lastly we mention the following result related to the genus function mentioned earlier.

Theorem 5.3. *[4, Theorem 1.4.17] Let X be a compact complex four-dimensional manifold, and $i : C \hookrightarrow X$ a smooth (non-singular) connected complex curve. Then we have*

$$2g(C) - 2 = [C]^2 - c_1(X)[C], \tag{5.2}$$

where $g(C)$ is the genus of C.

Perhaps later on we will need to recall some of the results in this section, but as we will mostly skip the section on gauge theory I chose to skip it for now.

References

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