<u>Today</u>: Genus function & SW theory.

Last week we saw ~~that~~ the inters. pairing on H^2 plays a fund. role in the theory of top. ~~m~~ 4-mfds. Today we'll focus more on a "smooth aspect" of the cohomology H^2. Recall from last week that <u>every</u> class in $H_2^{\#}(X;\mathbb{Z})$ is repr. by a closed embedded surface, oriented.
i.e. $\forall\ \alpha \in H_2(X;\mathbb{Z})$, $\exists\ i: \Sigma \hookrightarrow X$ $\quad i_*[\Sigma] = \alpha$.

Define the genus function

$$G: H_2(X;\mathbb{Z}) \longrightarrow \mathbb{Z}_{\geq 0}$$

by $\left(\cancel{G(0) = 0 \quad (\alpha = 0 \text{ follows from defn})}\right)$

$$G(\alpha) = \min\{\text{genus}(\Sigma) \mid \Sigma \text{ c.l.or. emb. surface repr. } \alpha\}$$

~~Note.~~ note: $\bullet\ G(0) = 0$

$\bullet\ G(-\alpha) = \alpha$.

1^{st} point also explains why we take the minimum:

adds 1 to g.

In general it's ~~only~~ ~~realist~~ a difficult function, often only $\leq$ estimates known. Our main goal today is

① $\bullet$ discuss G on $\mathbb{C}P^2$ (by blow-ups)

② $\bullet$ recap SW-theory and "prove" gen. Thom theorem.

①: recall: $H_2^{\#}(\mathbb{C}P^2) = \mathbb{Z}\cdot h$, where $h = [H]$,

$\mathbf{H} = \{[x:y:z] \mid x = 0\} \cong \mathbb{C}P^1$, hence $G(h) = 0$.

Now let $f \in \mathbb{C}[x,y,z]_d$ be a hom. pol. of degree d, s.t. $Z(\text{Jac}(f)) = (0,0,0)$ (so $\{f=0\} \subset \mathbb{C}P^2$ is smooth). ~~It's a fact that then~~
(E.g., $f = x^d + y^d + z^d$). ~~It's a fact that~~ Then $\{f = 0\} =: Z(f)$ is connected, and represents $d \cdot h$ in $H_2(\mathbb{C}P^2)$. <u>Pf</u>: $H \cap Z(f) = \{[0:y:z] \mid y^d + z^d = 0\}$

Say $f = x^d + y^d + z^d$ ...

What is genus of $Z(f)$? In general, let $C \hookrightarrow S$ be a cx-curve in a cx. mfd. Then $TS|_C = TC \oplus NC$

$$c_1(TS)[C] = c_1(TC)[C] + c_1(NC)[C]$$
$$= \chi(C) + [C]\cdot[C]$$
$$= 2-2g(C) + [C]\cdot[C].$$

facts: • $c_{top}(E) = e(E)$ (use splitting princ. top. cx.)

• $X \hookrightarrow Y$, $e(N_{X/Y}) = \eta_X|_X$, so $\int_X e(N_{X/Y}) = \int_X \eta_X = \int_Y \eta_X \wedge \eta_X = [X]\cdot[X]$,
(Bott–Tu)

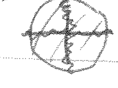
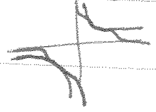
So for $\underbrace{Z(f)}_{C} \subseteq \mathbb{CP}^2$, $\underset{[C]}{\underset{\parallel}{[Z(f)]}} = d\cdot[H]$, $c_1(\mathbb{CP}^n) = (1+a)$ ~~$c_1(\mathbb{CP}^2) = PD(H)$~~ $c_1(\mathbb{CP}^2) = 3a$, $a = PD(H)$ ~~[H]~~ [H]

$3[H]\cdot[C] = 2-2g(C) + [C]\cdot[C]$

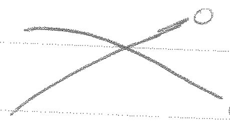
$3d = 2-2g(C) + d^2$

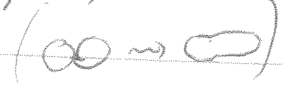
$$g(C) = \frac{d^2+2-3d}{2} = \frac{(d-1)(d-2)}{2}$$

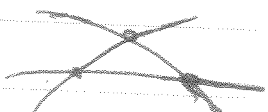
$\Longrightarrow G(dh) \leq \frac{(d-1)(d-2)}{2}$.

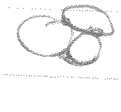
One might try to start with (e.g. sing. nodal) curves, and try to desingularize them. E.g., repr. $d\cdot h$ by d copies of $\mathbb{CP}^1$ in $\mathbb{CP}^2$, each inters. transversal at 1 pt. Such an intersection looks like $z_1 z_2 = 0$:  locally, and one deforms this into $z_1 z_2 = \varepsilon$: , also deforming the ends.

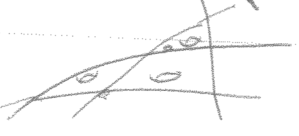
End result is htp to start, now it's smooth.

$d=2$  ⇝ genus 0



$d=3$  ⇝ genus 1



$d=4$  ⇝ genus 3

Next, relate this to blow ups.

$$\tau := \{(p,\ell)\in \mathbb{C}^2\times\mathbb{CP}^1 \mid p\in\ell\} \xrightarrow{\pi_1} \mathbb{C}^2 \quad (\text{bihol. of } \tau\setminus\tau_0 \text{ w/}$$

$$\downarrow \pi_2$$

$$\mathbb{CP}^1$$

note that $\pi_2^{-1}(\ell)=\ell$ as $\mathbb{C}$-v.sp., τ = taut. line bundle.

Since ~~...~~ $\mathbb{C}$-line bdles on $\mathbb{CP}^1 \cong H^2(\mathbb{CP}^1;\mathbb{Z})=\mathbb{Z}$, ~~with generator~~

τ is classified (~~diffeom.~~) by $\langle c_1(\tau),[\mathbb{CP}^1]\rangle$. To compute, consider

$\tau^* = \mathrm{Hom}(\tau, \mathbb{C}_{\mathbb{CP}^1})$ and ~~take the global section~~ consider the map

$\tau \to \mathbb{C}$

$\{(z_1,z_2,\ell)\mapsto z_1$. It's linear on fibers, hence gives rise to a section of τ^*. Its zero set is the point $\{[0:1]\in\mathbb{CP}^1$.

$\Rightarrow$ degree of $\tau^*=1$, i.e. $c_1(\tau^*)=1 \Rightarrow c_1(\tau)=-1$

(common termin.: $\tau^* =: \mathcal{O}(1)$, $(\tau^*)^{\otimes n} =: \mathcal{O}(n)$, also on $\mathbb{CP}^k$, $k\geq 1$)

Prop: τ is diffeom. to $\overline{\mathbb{CP}^2}\setminus\{p\}$, for any $p\in\mathbb{CP}^2$.

pf: ~~consider $\overline{\mathbb{CP}^2}$~~ Take a proj. line ($\cong\mathbb{CP}^1$) missing p, say L_0.
Then we have a map $\pi:\overline{\mathbb{CP}^2}\setminus\{p\}\to L_0$, $\pi(Q) := L_{pq}\cap L_0$

L_{pq} the unique proj. line through p & Q. Any line diff. from L_0

~~W.l.o.g. may assume P =~~ (fiber over Q is $L_{pq}\setminus p \cong \mathbb{C}$)

and not going through p gives a section of $\overline{\mathbb{CP}^2}\setminus p\to\mathbb{CP}^1$

whose int. w/ zero section gives -1 ($+1$ in $\mathbb{CP}^2$, hence -1 in $\overline{\mathbb{CP}^2}$)

Therefore as cx. line bdle (in fact, as hol. line bundle) we have $\overline{\mathbb{CP}^2}\setminus\{p\}\cong\tau$. Note that the "ends" of fibers in τ correspond to pts nearby p in $\overline{\mathbb{CP}^2}$.

Illustr.: $P=[0:0:1]$, $L_0=\{z=0\}$. If $Q=[a:b:c]$,
$\pi(Q)=[a:b]$. Fiber over $[a:b]$ is $\{[a:b:c]\mid c\in\mathbb{C}\}$
$\cong\mathbb{C}$.

□

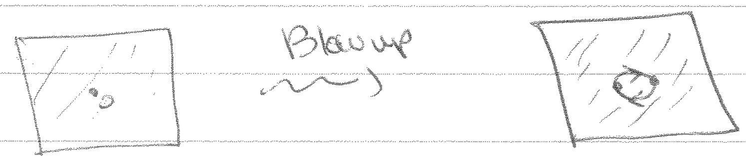
now consider ~~$\mathbb{C}$~~ $\tau \xrightarrow{\pi} \mathbb{C}^2$

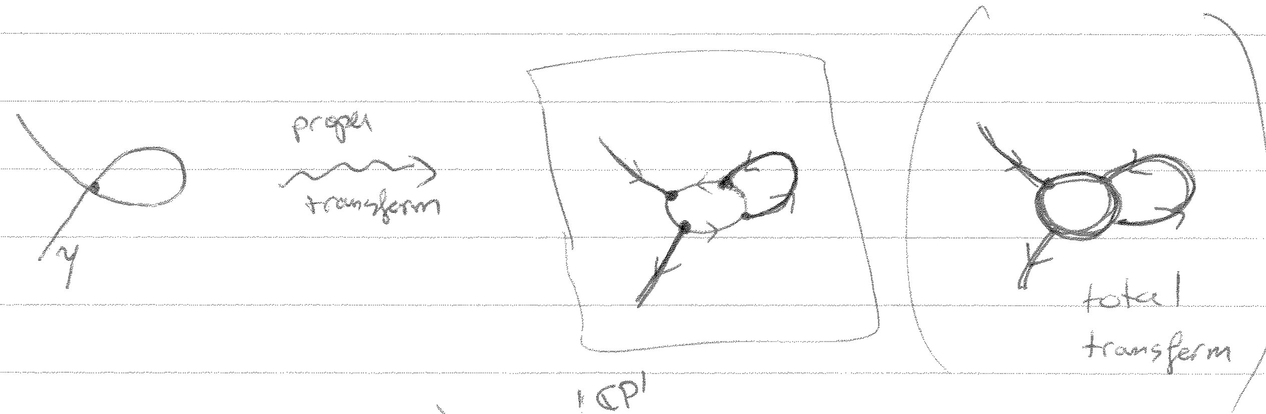
~~easy~~ readily verified: π is a bihol. map between

$$\underbrace{\tau - \pi^{-1}(0)}_{\{(p,\ell) \mid p\in\ell,\, p\neq 0\}} \to \mathbb{C}^2 \setminus \{0\}, \text{ while } \pi^{-1}(0) = \mathbb{CP}^1$$

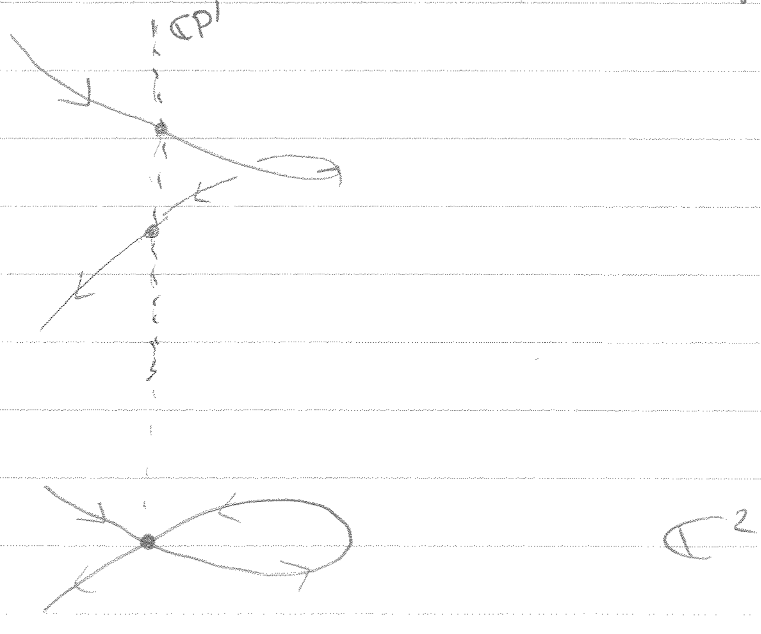
note that if L_1, L_2 are curves in $\mathbb{C}^2$ meeting just at 0 transversally, their socalled proper transforms $\tilde{L}_i := \pi^{-1}(L_i - \{0\})$ are disjoint.

~~Thi~~ (there is also the total transform $L_i' := \pi^{-1}(L_i)$)



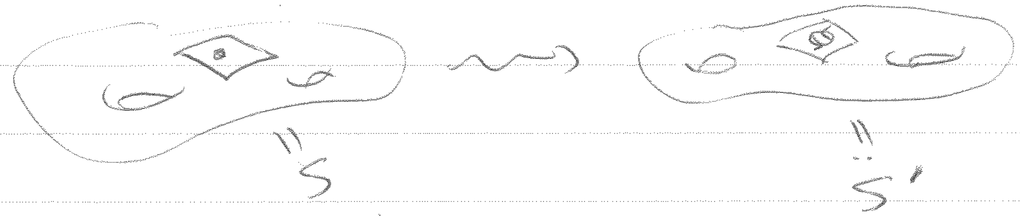


other picture:



The $\mathbb{C}P^1 = \tau_0$ is called the <u>exceptional divisor</u>

Now if S is any cx. surface, $p \in S$ any pt., pick local hol. chart $p \in \mathcal{U} \cong \mathbb{C}^2$, can blowup at 0 in the chart



FACT: resulting cx. mfd structure on S' is indep. of chosen chart. From our description of $\tau = \overline{\mathbb{C}P^2} \setminus pt.$, we see that

$$S' \overset{\text{diff.}}{\cong} S \# \overline{\mathbb{C}P^2}$$

If X is any smooth or. 4-mfd, $X' := X \# \overline{\mathbb{C}P^2}$ is the blowup of X, $\overline{\mathbb{C}P^1} \hookrightarrow \overline{\mathbb{C}P^2}$ is the exc. sphere, $e := [\overline{\mathbb{C}P^1}]$ its homcl.

Note, by Mayer-Vietoris

$$H_2(X') \cong H_2(X) \oplus H_2(\overline{\mathbb{C}P^2}) \cong H_2(X) \oplus \mathbb{Z} \cdot e \qquad (*)$$

$$Q_{X'} = Q_X \oplus \langle -1 \rangle \qquad (e^2 = -1)$$

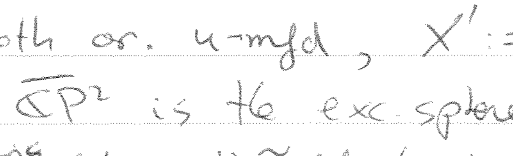
If $\Sigma \subseteq X$ a (cx) surface, $X' \xrightarrow{\pi} X$ blow up in $p \in X$, then

$\Sigma' := \pi^{-1}\Sigma$ tot. transf. (only interesting if

$\tilde{\Sigma} := \overline{\pi^{-1}(\Sigma \setminus p)}$ prop. " " $p \in \Sigma$)

observe that $[\tilde{\Sigma}] = [\Sigma] - e$ in (*)

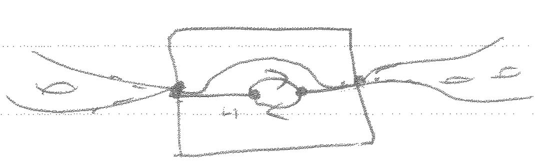
<u>geom. proof:</u>



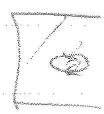
$\Sigma \simeq \Sigma$

Blow up

$L_1 \cdot \mathbb{C}P^1 = 1$ in $\mathbb{C}P^2 \setminus \{pt\}$. conj. twice.



$\tilde{\tilde{\Sigma}} \ncong \tilde{\Sigma} \cong \Sigma$ (*)

$\tilde{\tilde{\Sigma}} - \tilde{\Sigma}$ is a class in 

so is a mult. of e

$\cancel{\lambda} \neq [\tilde{\Sigma}] \cdot e = -\lambda$

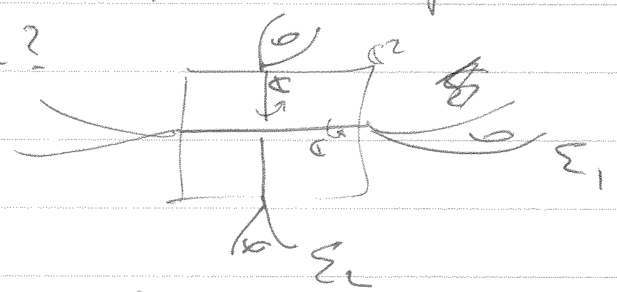
-1

$[\tilde{\tilde{\Sigma}}] = [\Sigma] + \lambda \cdot e$

So if we have Σ_1 & Σ_2 $\pitchfork$ at p, get

$$[\tilde{\Sigma}_1] = [\Sigma_1] - e \qquad [\tilde{\Sigma}_2] = [\Sigma_2] - e \quad \Rightarrow [\tilde{\Sigma}_1]\cdot[\tilde{\Sigma}_2] = [\Sigma_1][\Sigma_2] - 1$$

__1 caveat:__ How do we know we can put this in the local model?



Need: $(\Sigma_1 \cdot \Sigma_2)_p = +1$

Can understand this geometrically: taking conn. sum

X w/ $\overline{\mathbb{C}P^2}$ amounts to:

$p \in X$, $D \ni p$ little disk, $q \in \overline{\mathbb{C}P^2}$, $D' \ni q$ little disk.

$$D \underset{\text{or. rev.}}{\overset{\sim}{\longleftrightarrow}} D'$$

want: $\Sigma_1, \Sigma_2 \subseteq D \leftrightsquigarrow$ cx. lines $L_1, L_2 \subseteq D'$ also or. rev. Comp. requires (rem. $L_1 \cdot L_2 = -1$ so L_1, L_2 neg. or. of D') $\Sigma_1 \cdot \Sigma_2 = +1$ at p.

$$(-1 \cdot -1 = 1 = -(-1))$$

If $(\Sigma_1 \cdot \Sigma_2)_p = -1$, can resolve in 2 ways. Either let one of L_1 or L_2 reverse: $L_1 \leftrightsquigarrow \overline{L_1}$, or take $\mathbb{C}P^2$ over $\overline{\mathbb{C}P^2}$ (reason $\overline{\mathbb{C}P^2}$ better than $\mathbb{C}P^2$ is that for S cx., $S \# \mathbb{C}P^2$ is not even almost cx.) In both cases $([\hat{\Sigma}_1]\cdot[\hat{\Sigma}_2]) = [\Sigma_1][\Sigma_2] + 1$

The "reverse process" of all this is <u>Blowing down</u>

Prop: suppose $\mathbb{C}P^1 \cong C \underset{\text{emb.}}{\overset{\text{hol.}}{\hookrightarrow}} \underbrace{S}_{\text{cx. surface}}$ ("say: S has a rat. curve")

~~then~~ with $[C]^2 = -1$. Then $\exists$ cx. surf. T s.t. $S \underset{\text{(bihol.)}}{\cong} T' =$ blow-up of T, s.t. $C =$ exc. curve.

(say: "C can be contracted in S")

Prop: If X (4 mfd) contains a sphere Σ_- (resp. Σ_+) s.t. $\Sigma_-^2 = -1$ (resp. $\Sigma_+^2 = +1$) then $\exists$ 4 mfd Y s.t. $X = Y \# \overline{\mathbb{C}P^2}$ (resp. $X = Y \# \mathbb{C}P^2$)

The copy $\overline{\mathbb{C}P^2}\setminus p$ (resp $\mathbb{C}P^2\setminus p$) can be chosen a tub. nbhd of Σ_- (resp Σ_+).

pf (for Σ_+): take $\underbrace{N_{\Sigma_-/X}}_{\text{normal bdle.}} \hookrightarrow X$ as tub. nbd of Σ_- in X.

$$[\Sigma_-]^2 = e(N)[\Sigma] = -1$$

$$\Sigma_- \cong \mathbb{C}P^1 \Rightarrow N \underset{\text{c.v.}}{\cong} \tau \cong \overline{\mathbb{C}P^2}\setminus\{p\}.$$

$$\partial(X \setminus N) = S^3, \text{ take } Y = (X\setminus N)\cup_{S^3} D^4 \qquad \square$$

Thm: $G(3h) = 1$ in $\mathbb{C}P^2$. (know already $G(3h) \leq 1$)

pf: enough to show that $3h \neq [\Sigma]$, $\Sigma \cong \mathbb{C}P^1$.

Suppose on contrary it does

$$[\Sigma^2] = 9 \cdot h^2 = 9.$$

may assume Σ has only transv. self int. pts, then blowing up 8 times gives $[\underbrace{\tilde{\Sigma}}_{\cong \mathbb{C}P^1}]^2 = +1$ in $\mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}$

if $e_1 \ldots e_8$ exc. spheres, $[\tilde{\Sigma}] = 3h - \sum_{i=1}^{8} e_i$

appl. prev. prop., write $\mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2} = Y \# \mathbb{C}P^2$

note that $H_2(Y \# \mathbb{C}P^2) = H_2(Y) \underset{\perp \text{ for pairing.}}{\oplus} \mathbb{Z}\cdot[\tilde{\Sigma}]$

Then $\{e_1-e_2, e_2-e_3, \dots, e_7-e_8, e_6+e_7+e_8-h\} \subseteq H_2(Y;\mathbb{Z})$ since all these are perp. to $[\Sigma]$. Letting $\mathbf{A} = \langle \dots \rangle$ be the $\mathbb{Z}$-subm., $Q|_A \cong -E_8$ is unimodular. By lemma 1.2.12 (cor 1.2.13) $A = H_2(Y;\mathbb{Z}) \Rightarrow Y$ is a smooth s.c. closed 4 mfd w/ $\sigma = -8$ $\lightning$ by Rochlin's thm. (Also by Donaldson's theorem??) $\square$

We want to "prove" the following result which makes life easier on cx. surfaces

$\boxed{\text{Gen. Thom thm}}$: Let S be a s.c. cl. cx. surface, $C \hookrightarrow S$ a cx smooth curve. If $C' \hookrightarrow S$ is a real oriented smooth surface repr. same hom. class as C, then $g(C') \geq g(C)$.

COR: $G(d \cdot h) = \frac{1}{2}(|d|-1)(|d|-2)$ on $\mathbb{CP}^2$

To see where this result comes from, have to recap SW-theory.

SW-theory

In 4d, have a nice descr. of Spin & Cliff. (Cl_n = alg. gen. on $e_1 \dots e_n$, $e_ie_j + e_je_i = -2\delta_{ij}$.)

$$\left(Spin(4) \times_{\mathbb{Z}_2} U(1) \cong\right) Spin^c(4) \hookrightarrow \mathbb{C}l_4^0 \subseteq \mathbb{C}l_4 \underset{\text{class.}}{\cong} \mathbb{C}(4)$$

$$\mathbb{C}l_4^0 \cong \mathbb{C}(2) \oplus \mathbb{C}(2)$$

$$\mathbb{C}l_4 \curvearrowright S \underset{\text{v.s.}}{\cong} \mathbb{C}^4$$

under $Spin^c(4)$, $S \overset{*}{\cong} \underset{\mathbb{C}^2}{S^+} \oplus \underset{\mathbb{C}^2}{S^-}$ $\quad (\mathbb{C}l^1 : S^\pm \to S^\mp)$

use this to show that $Spin^c(4) \cong S(U(2) \times U(2)) = \{(A,B) \mid \det A = \det B\}$

$$U(1) \xleftarrow{\det} Spin^c(4) \xrightarrow{\rho} SO(4)$$

$$\lambda^2 \longleftarrow [(g,\lambda)] \longmapsto \rho(g)$$

$*$: decomp. is a ± 1 eig.sp. decomp. for $\omega := -e_1e_2e_3e_4$

Riem. 4-fd

call (X,g) a Spin^c mfd if $\exists$ a principal Spin$^c(4)$ bdle $\tilde{P}$, together with an ~~double cover~~ equiv. map $\tilde{P} \to P := P_{SO(4)}$ ass. to g.

(equiv. w.r.t. $Spin^c(4) \to SO(4)$)

$\boxed{\text{~~Thm~~} \; \exists \text{ Spin}^c\text{-str. on every or. 4-mfd}}$

Given $\tilde{P}$, the hom. $Spin^c \xrightarrow{\det} U(1) \rightsquigarrow L = \tilde{P} \times_{Spin^c} \mathbb{C}$ called det. line bdle. It satisfies $c_1(L) \equiv w_2 \mod 2$, (so is char.) and conversely any L w/ $c_1(L) \equiv w_2 \mod 2$ det. a $\tilde{P}$ s.t. L is the ass. line bdle. So $S^c(X) \overset{1:1}{\longleftrightarrow} H^2(X;\mathbb{Z})$, and if $\exists$ no 2-tors. in H^2, also $S^c(X) \overset{1:1}{\underset{\text{non-can.}}{\longleftrightarrow}}$ char. elts $= \{K \in H^2(X;\mathbb{Z}) \mid K \equiv w_2 \mod 2\}$

$\boxed{\text{Thm (CHS): for any } X^4 \text{ char. elts} \neq \emptyset \Rightarrow \exists \text{ Spin}^c\text{-structure}}$

Ass. to the 2 fund. rep^n of $Spin^c(4)$ are 2 spinor bundles $S^\pm := \tilde{P} \times_{Spin^c(4)} S^\pm$, $S := S^+ \oplus S^-$ is a Clifford module for $\mathbb{C}l(X,g)$. Ass. to the L.C. conn. on TX & a unit. conn. A on L is a ~~Dirac operator~~ conn. on $S^\pm$, hence a Dirac operator $\partial\!\!\!/_A \left(= \sum_{i=1}^4 e_i \cdot \nabla^A_{e_i}\right) = \begin{pmatrix} 0 & \partial\!\!\!/_A \\ \partial\!\!\!/_A & 0 \end{pmatrix} : S \to S$.

Finally, note that $\exists!$ metrics $\langle , \rangle$ on $S^\pm$ inv. under Spin^c, so that given $\psi \in \Gamma(S^+)$ get an end. $\Gamma(S^+) \to \Gamma(S^+)$ given by $q(\psi) := \psi \otimes \langle -, \psi \rangle - \underbrace{\tfrac{1}{2}\langle \psi, \psi \rangle \cdot Id}_{\text{make it traceless.}}$

Finally, note that $\Lambda^\bullet T^*_{\mathbb{C}} \cong \mathbb{C}l(X,g) \cong \underbrace{End_{\mathbb{C}}(S,S)}_{\mathbb{C}(4)}$

$e_{i_1} \wedge \dots \wedge e_{i_j} \mapsto e_{i_1} \cdot \dots \cdot e_{i_j}$ $\quad \mathbb{C}(4)$

can check directly: $\Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4 \cong End(S^+) \oplus End(S^-)$

$\Lambda^1 \oplus \Lambda^3 \cong Hom(S^+, S^-) \oplus Hom(S^-, S^+)$

$1 \in \Lambda^0$ acts as $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$, $\omega := e_1 \cdot \dots \cdot e_4$ ($e_1 \wedge \dots \wedge e_4$) is the vol. element, $S^\pm$ are ± 1 eig.sp., so $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow \Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ as tr. free end. of S^+ & S^-.

easy: $\Lambda^2_- \circlearrowright S^+$, $\Lambda^2_+ \circlearrowright S^-$ as 0. $((e_1 e_2 + e_3 e_4) \cdot \psi = -(\dots)\cdot\psi)$ $\quad \psi = \omega\psi$

$$\underset{\text{tr. free}}{\mathrm{End}(S^+)} \cong \Omega^2_+(X, \mathbb{C})$$

check: (using $\langle e\cdot\psi, \varphi\rangle = -\langle \psi, e\varphi\rangle$)
that $\Omega^2_+(X, i\mathbb{R})$ acts as herm. end.

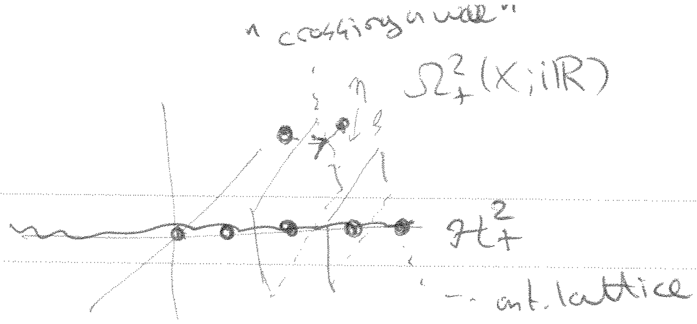
So finally, it makes sense for $A \in \mathcal{A}(L)$ = unit. conn. on L, $\psi \in \Gamma(S^+)$ and $\eta \in \Omega^2_+(X; i\mathbb{R})$ to write down

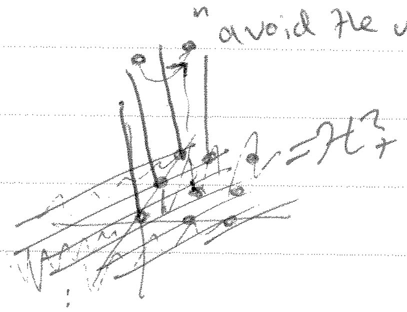
$$F_A^+ = q(\psi) + \eta \qquad \text{(monopole)}$$
$$\partial\!\!\!/_A \psi = 0 \qquad \text{(Dirac)}$$

We're in the set up:
$\mathcal{C} = \mathcal{A}(L) \times \Gamma(S^+)$ ("config. space")
$\mathcal{P} = \Omega^2_+(X; i\mathbb{R})$ ("parameter space")
$\mathcal{G} = \mathrm{Map}(X, U(1))$ ("gauge group")
$F: \mathcal{C} \times \mathcal{P} \to \Omega^2_+(X, i\mathbb{R}) \times \Gamma(S^-)$ ("eqn")

<u>facts</u>:
- If $b_2^+ > 0$, for generic η $\nexists$ red. sol^n ($\psi = 0$) to $F_\eta(A, \psi) = 0$, and the perturbed moduli space $\mathcal{M}_\eta$ is smooth, compact, orientable, of dim $d = \frac{1}{4}(c_1(L)^2 - (2\chi + 3\sigma))$. orient. det. by orient. on H^0, H^1, H^2_+
- For simply conn. X, $\mathcal{B}^* := \mathcal{C}^*/\mathcal{G}$ is a class. space for $\mathrm{Map}(X, U(1)) \sim_{h.t.} U(1)$, so $\mathcal{B}^* \underset{h.t.}{\cong} \mathbb{CP}^\infty$, $H^*(\mathcal{B}^*; \mathbb{Z}) = \mathbb{Z}[\mu]$ μ degree 2,
$$SW(\tilde{\mathfrak{p}}) := \int_{\mathcal{M}_\eta} \mu^{d/2} \quad (\eta \text{ small pert.})$$
for $b_1^+ > 1$, this is indep. of g or η.
for $b_1^+ = 1$ there are "wall-crossing" formulas.

<u>remark</u>: $((A, 0), \eta)$ red. sol^n for $\mathcal{M}_\eta$ if $F_A^+ = \eta$ in $\Omega^2_+(X; i\mathbb{R})$
proj. this eqn to $\mathcal{H}^2_+ \hookrightarrow \Omega^2_+(X; i\mathbb{R}) = \mathcal{H}^2_+ \oplus \mathrm{Im}\, d \oplus \mathrm{Im}\, d^*$
$\mathcal{H}^2_+ \cong H^2_+(X; \mathbb{R})$. Since $\frac{i}{2\pi}[F_A] = c_1(L)$, a necessary condition for a red. sol^n is $\frac{i}{2\pi} \cdot \eta = c_1$ in $\mathcal{H}^2_+$, $c_1 \in H^2(X; \mathbb{Z}) \subseteq H^2(X; \mathbb{R})$





~~Sketches~~:

VANISHING

- $X=X_1\#X_2$ $b_2^+(X_i)>0 \Rightarrow SW_X=0$
- $\exists g$ on X s.t. $s_g>0 \Rightarrow SW_X=0$.
- $\Sigma\hookrightarrow X$ emb. sphere $[\Sigma]^2\geq 0$, $[\Sigma]\neq 0$, $SW_X=0$

NON VANISHING

- S simply conn. cx. surf., $b_2^+(S)>1$ (so it's odd) then $SW_S(\pm c_1(S))\neq 0$
- (more generally) (X,ω) simply conn. sympl. & $b_2^+>1$
  $SW_X(\pm c_1(X,\omega))=\pm 1$.

Now comes the result of importance to us. We say that a characteristic element K is <u>basic</u> if $SW_X(K)\neq 0$, and that X is of <u>simple type</u> if $\forall$ basic K, $M_\eta(K)$ is 0-dim. (only non-triv. inv. come from 0-d. mod. sp.)

<u>Generalized Adjunction formula</u>:

assume $\Sigma \subseteq X$ ($b_2^+ > 1$) emb. or. conn. surf., $g(\Sigma) =$ genus, $[\Sigma]^2 \geq 0$.
Then $\forall\, K \in \mathrm{Bas}_X$, $2g(\Sigma) - 2 \geq [\Sigma]^2 + |K[\Sigma]|$

<u>If</u> X is of simple type and $g(\Sigma) > 0$, same expr. is true ~~even~~ for all Σ (so even if $[\Sigma]^2 \leq 0$).

<u>Pf of gen. Thom conj.* for $b_2^+>1$</u>: S cx. s.c. surface $\overset{\text{thm}}{\Longrightarrow}$ S is Kähler hence of simple type. Since $c_1(S) \in \mathrm{Bas}_X$, we get

$$2g(\Sigma) - 2 \geq [\Sigma]^2 + \cancel{\underbrace{|c_1(S)[\Sigma]|}_{[\Sigma]^2 + 2 - 2g(\Sigma)}} \;\cancel{\geq [\Sigma]^2 + [\Sigma]^2 + 2g(\Sigma}$$

$$= [C]^2 + \underbrace{|c_1(S)[C]|}_{[C]^2 + 2 - 2g(C)} \geq 2g(C) - 2 \qquad \square.$$