3 Complex surfaces

The goal:
- give examples of 4-manifolds
- show how new examples ~~are~~ can be constructed from old ones (fibre sum + logarithmic trans.)
- ~~[illegible]~~ construct all simply connected complex surfaces not of general type (as C^∞ manifolds)

Rk: We're mostly interested in the diffeo. type of a complex surface.
homeo/

Rk: This talk will be devoid of proofs.
(Many are found in chapter 7/8 or use SW-theory.)

Today:
- S is a compact complex surface (often $\pi_1(S) = 0$)
- C is a compact complex curve.

Overview:

3.1 The manifolds $E(1)$ - elliptic surfaces
3.2 $E(n)$ - fibre sums
3.3 $E(n)_{p,q}$ - logarithmic transformations
3.4 Classification

3.1 The manifold E(1).

dfn: S is an elliptic surface if it admits a ~~proper~~ holomorphic map to a curve C,

S is cpt $\pi : S \longrightarrow C$,

s.t. the generic fibre $\pi^{-1}(t)$ is a C^∞ elliptic curve.

π is then called a holomorphic elliptic fibration.

dfn: A smooth map $\pi : X^4 \longrightarrow C$ is called a smooth elliptic fibration if every $x \in X^4$ has a neighbourhood that can be embedded in an elliptic surface (compatible w/ π and the orientation,

$$\begin{tikzcd} \overset{t\in}{X} \supseteq U \arrow[r, hook, "\varphi"] \arrow[dr, "\pi"'] & S \arrow[d, "\pi'"] \\ & C \end{tikzcd}$$

e.g. $E \times C$ for E an elliptic curve and C any curve.

e.g. E(1) (and manifolds constructed from it)

Let p_0, p_1 be generic cubic polynomials on $\mathbb{C}^3$ ($\gcd(p_0, p_1)=$ and let $V_{p_i} := \{[x:y:z] \in \mathbb{C}P^2 \mid p_i(x,y,z) = 0\} \subseteq \mathbb{C}P^2$

Now $[V_{p_i}] = 3 \in H_2(\mathbb{C}P^2) \simeq \mathbb{Z}$, so $[V_{p_0}] \cdot [V_{p_1}] = 9$.

Call the 9 intersection points $P_1, P_2, \dots, P_9 \in V_{p_0} \cap V_{p_1}$.

We'll consider the family

$(V_t)_{t \in \mathbb{C}P^1}$, where $V_{[t_0:t_1]} = \{x \in \mathbb{C}P^2 \mid t_0 p_0(x) + t_1 p_1(x) = 0\}$

These all *intersect transversally* in $P_1, \dots, P_9$, and nowhere else (since $[V_t] \cdot [V_{t'}] = 9$ still holds)

Hence, for every $x \in \mathbb{C}P^2 \setminus \{P_1, \dots, P_9\}$ there is a *unique* $t \in \mathbb{C}P^1$ s.t. $x \in V_t$ and we obtain a map

$$\pi : \mathbb{C}P^2 \setminus \{P_1, \dots, P_9\} \longrightarrow \mathbb{C}P^1 \quad (\text{holomorphic})$$

s.t. $\pi^{-1}(t) = V_t \setminus \{P_1, \dots, P_9\}$.

Since V_t meet transversally at $P_1, \dots, P_9$, ~~so~~ they parametrise lines through these points and *we can extend π* to

$$\pi : \mathbb{C}P^2 \# 9 \overline{\mathbb{C}P^2} \longrightarrow \mathbb{C}P^1$$

by *blowing* these points *up*.

dfn: This is the manifold $E(1) = \mathbb{C}P^2 \# 9 \overline{\mathbb{C}P^2}$

Rk:
- The fibres of $E(1) \to \mathbb{C}P^1$ are *cubic curves*, hence *elliptic* ($g = \frac{1}{2}(d-1)(d-2) = 1$ if $d = 3$)
- The generic fibre is smooth, so *E(1) is an elliptic surface*

lem:
- $\underline{\pi_1}(E(1)) \underline{= 0}$
- $\underline{b_2}(E(1)) \underline{= 10}$ and $\underline{Q_{E(1)} \simeq \langle 1 \rangle \oplus 9 \langle -1 \rangle}$

  $\Rightarrow \underline{\chi}(E(1)) = \underline{12}$, $\underline{\sigma}(E(1)) = \underline{-8}$,
- $c_1(E(1))^2 = 0$, $c_2(E(1)) = 12$
- $c_1(E(1)) = 3h - \sum_{i=1}^{9} e_i$

  $\uparrow$ generator of $H_2(\mathbb{C}P^2)$ (under h); $\uparrow$ exceptional spheres (beside e_i)

dfn: A *fish-tail fibre* is one that is isotopic to

$$C_1 = \{zy^2 = x^3 + zx^2\} \subseteq \mathbb{C}P^2 \qquad (\sim \alpha)$$

A *cusp fibre* is isotopic to

$$C_2 = \{zy^2 = x^3\} \subseteq \mathbb{C}P^2 \qquad (\sim \prec)$$

N.B. $\chi(C_1) = 1$ & $\chi(C_2) = 2$.

thm: If $\varphi : X \twoheadrightarrow C$ is proper, holomorphic ~~the~~ with generic fibre F & singular fibres $F_1, \dots, F_N$ then

$$\chi(X) = \chi(C)\chi(F) + \sum_{i=1}^{N} \big(\chi(F_i) - \chi(F)\big)$$

cor: $\sum_i \chi(F_i) = \cancel{12}\chi(E(1)) = 12$ for F_i the singular fibres of π.

prp:
- For p_0, p_1 generic, π has 12 fish-tail fibres
- p_0, p_1 can be chosen s.t. π has at least one cusp.
  ($p_0 = zy^2 - x^3$, $p_1 = x^3 + y^3 + z^3$)

Rk: All possibilities have been classified by Kodaira.

Rk: Can do this for polynomials of any degree d, to obtain a "fibration"

$$\pi : \mathbb{C}P^2 \# d^2 \overline{\mathbb{C}P^2} \longrightarrow \mathbb{C}P^1$$

with generic fibre a C^∞ curve of genus $g = \frac{1}{2}(d-1)(d-2)$

3.2 Fibre sums & E(n)

Let $\pi_i : S_i \xrightarrow[C^\infty]{} C_i$ be two elliptic ~~surfaces~~ filrations $(i=1,2)$,
~~[illegible] $\pi_i^{-1}(t)$ are regular fibres~~
let $\underline{\Delta_i} \subseteq S_i$ be diffeo to D^2 s.t. the fibres $\pi_i^{-1}(t)$ are regular for $t \in \Delta_i$.

Now $v_i := \pi^{-1}(\Delta_i) \cong D^2 \times T^2$ and $\partial v_i \cong S^1 \times T^2 \cong T^3$.

Let $\underline{\varphi} : \partial v_1 \xrightarrow{\sim} \partial v_2$ be fibre preserving + orient. reversing.

dfn: $\underline{S_1 \#_f S_2 := (S_1 \setminus v_1) \sqcup_\varphi (S_2 \setminus v_2)}$
is the fibre sum of S_1 and S_2
It comes with an elliptic filration $S_1 \#_f S_2 \longrightarrow C_1 \# C_2$.

~~Rk:~~
Rk: $S_1 \#_f S_2$ does not depend on φ if S_1 or S_2 has a cusp fibre.

dfn: $E(n) := \underbrace{E(1) \#_f E(1) \#_f \cdots \#_f E(1)}_{n}$

This spoils the complex structure
Rk: $E(n)$ can be viewed as an ~~[illegible]~~ elliptic surface.

$$E(n) \overset{C^\infty}{\cong} V(n) = \{ p \in \mathbb{C}P^1 \times \mathbb{C}P^2 \mid P_n(p) = 0 \} \xrightarrow{pr_1} \mathbb{C}P^1$$

for $\underline{P_n}$ a generic polynomial of $\overset{\wedge}{\text{bi-}}$ degree $(n,3)$ on $\mathbb{C}^2 \times \mathbb{C}^3$.

This complex structure is not unique.

lem:
$\pi_1(E(n)) = 0$
$\chi(E(n)) = 12n$
$b_2(E(n)) = 12n - 2$
$b_2^+ = 2n - 1$
$\sigma = -8n$
$c_1(E(n)) \overset{PD}{=} (2-n)[F] \rightsquigarrow c_1^2 = 0$
$E(n)$ is spin (Q even) $\Longleftrightarrow$ n is even.

<u>cor:</u> $E(2)$ is a K3 surface $(\pi_1 = c_1 = 0 \Leftrightarrow K3)$

<u>thm:</u> The SW basic classes of $E(n)$ are

$$\{PD(k[F]) \mid k \equiv n \pmod 2,\ |k| \le n-2\}$$

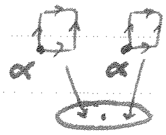
<u>thm:</u> $$Q_{E(n)} \simeq \begin{cases} n(-E_8) \oplus (2n-1)H & \text{if } n \equiv 0\ (2) \\ (2n-1)\langle 1\rangle \oplus (10n-1)\langle -1\rangle & \text{if } n \equiv 1\ (2) \end{cases}$$

3.3 Logarithmic Transformations (intr by Kodaira)

Let $\pi : S \to C$ be an *elliptic surface*

- $\Delta \subseteq C$ be s.t. $\Delta \overset{hol}{\cong} D^2 \subseteq \mathbb{C}$ & ~~[illegible]~~ $\pi^{-1}(\Delta) \to \overset{\Delta}{\cancel{C}}$ is regular

Rk: This means that the fibres are complex tori.
Pick a section $\alpha : \Delta \to \pi^{-1}(\Delta) =: \Phi$ ~~then~~ and declare $\alpha(z)$ to be the unit element in $\pi^{-1}(z)$,
then $\pi : \Phi := \pi^{-1}(\Delta) \to \Delta$
looks like $V/\Lambda \to \Delta$

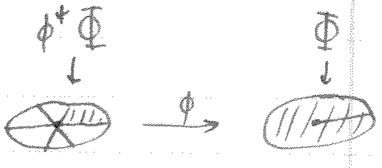


for V a ~~[illegible]~~ line bundle
and Λ a bundle of lattices in V.

The plan: to cut out Φ and glue a twisted version back in.

Let $p \in \mathbb{N}$

Consider $\phi_p : \Delta \to \Delta$, $z \mapsto z^p$



and let $\tilde{\Psi}_p := \phi_p^* \Phi = \{(z,v) \mid z \in \Delta, v \in \pi^{-1}(z^p)\}$

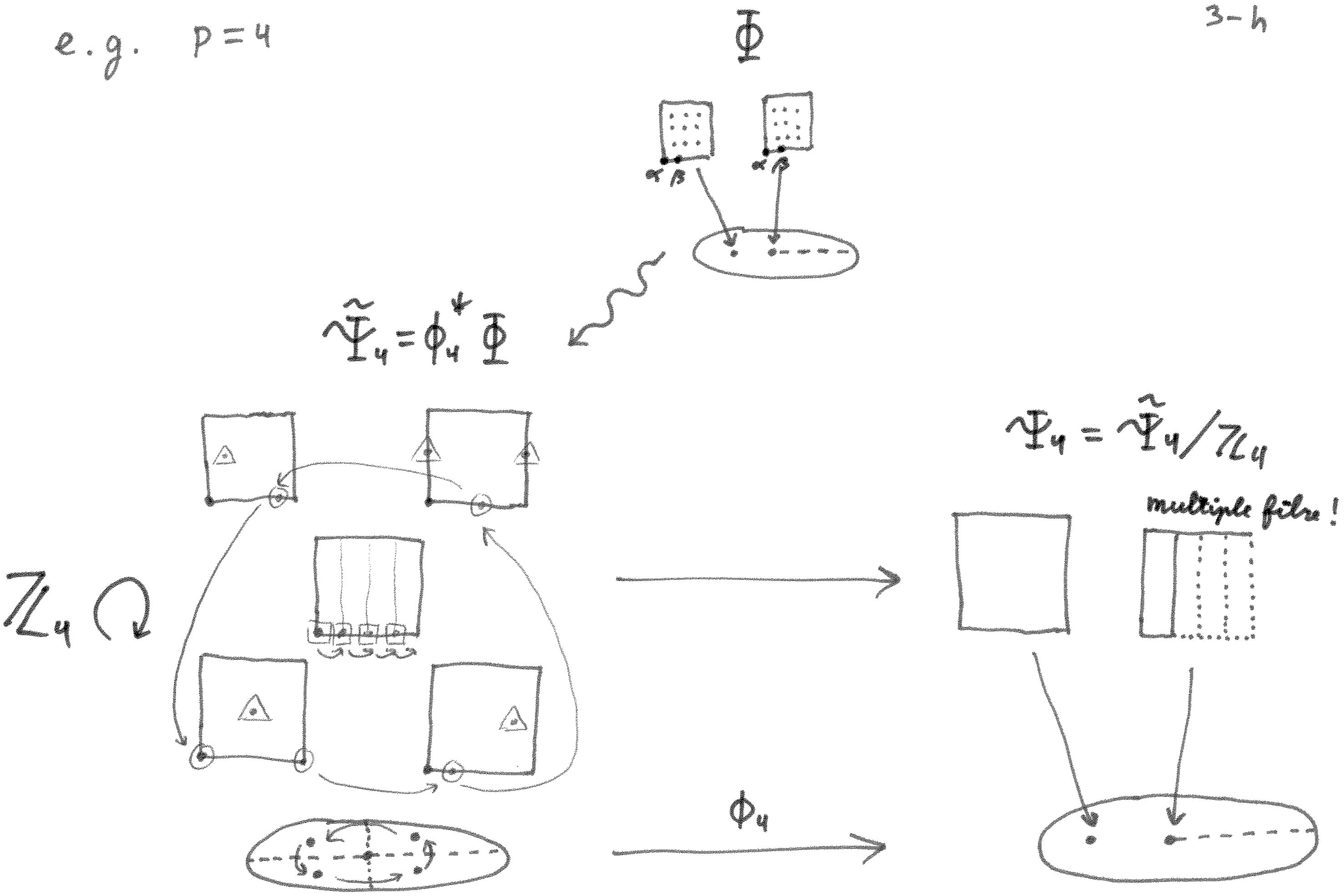
Now let $k \in \mathbb{Z}_p$ act on $\tilde{\Psi}_p$ as
$$k \cdot (z,v) = (\zeta_p^k z, v + \beta(z^p)\}$$
where $\beta : \Delta \to \Phi$ is any *section* consisting of elements *of order p* in $\pi^{-1}(z) = V_z/\Lambda_z$.

This action is free, so $\Psi_p := \tilde{\Psi}_p / \mathbb{Z}_p$ is again a *complex surface*.

The map $\psi : \Psi_p \to \Delta$, $[(z,w)] \mapsto z^p$ defines an *elliptic fibration*.

Rk: $\psi^{-1}(0)$ is a *multiple fibre*.
$p[\psi^{-1}(0)] = [\psi^{-1}(t)]$ ~~[illegible]~~ $\in H_2(\Psi)$ for $t \in \Delta \setminus 0$.

e.g. $p=4$


Φ
α β
α β
$\tilde{\Psi}_4 = \phi_4^* \Phi$
$\mathbb{Z}_4 \circlearrowright$
$\Psi_4 = \tilde{\Psi}_4 / \mathbb{Z}_4$
multiple fibre!
ϕ_4

Rk: $\Psi_1 = \Phi$

prp: $\lambda : \Psi_p \setminus \psi^{-1}(0) \longrightarrow \Phi \setminus \pi^{-1}(0)$

$[(z,v)] \longmapsto (z^p, v + \frac{2\pi i q}{p}] \log(z) \beta(z^p)$

is biholomorphic and

cor: Ψ_p and Φ coincide outside $0 \in \Delta$.

since

$$\begin{tikzcd}
\Psi_p \setminus \psi^{-1}(0) \arrow[rr, "\lambda"] \arrow[dr] & \arrow[d, phantom, "\circlearrowright"] & \Phi \setminus \pi^{-1}(0) \arrow[dl] \\
& \Delta \setminus 0 &
\end{tikzcd}$$

commutes.

This means we can cut out Φ and replace it by Ψ_p to obtain $S_p = S \setminus \pi^{-1}(0) \cup_\lambda \Psi_p = S \setminus \Phi \cup_{\lambda|_{\partial\Psi_p}} \overline{\Psi_p}$

dfn: S_p is the logarithmic transformation of p along $\pi^{-1}(t)$ for $t = 0 \in \Delta \subseteq S$.

Rk: If $\chi(S) \neq 0$ then S_p does not depend on Δ, α or β (as a smooth manifold)

Exercise 3.3.1: $\Psi_p \overset{C^\infty}{\simeq} T^2 \times D^2$,

so we're actually gluing the same manifold

[redacted] back in! [redacted]

We may define a smooth version of this procedure:

Let $t \in C$, $F = \pi^{-1}(t)$ and $\nu F \overset{C^\infty}{\simeq} T^2 \times D^2$, so $\partial\nu F \simeq T^2 \times S^1$.
Let $\varphi : T^2 \times S^1 \longrightarrow T^2 \times S^1$ be a diffeo, then

[redacted]

$\pi \circ \varphi : T^2 \times S^1 \xrightarrow{\varphi} T^2 \times S^1 \xrightarrow{\pi} S^1$

maps $[S^1] \in H_1(T^2 \times S^2)$ to some $p \in H_1(S^1) \simeq \mathbb{Z}$.

dfn: $S_\varphi := S \setminus \nu F \cup_\varphi \nu F$ is a logarithmic transform of multiplicity $|p|$.

thm: If S has a cusp fibre then S_φ only depends on $|p|$.

By repeating this process we obtain new manifols

$$S_{p_1, p_2, \dots, p_k} \qquad (p_i \geq 0)$$

Gompf

lem: · $\pi_1(\overset{E(n)}{}_{p_1 \dots p_k}) = 0$ iff $k \leq 1$ or $k = 2$ and $\gcd(p_1, p_2) = 1$

· If $\gcd(p,q) = 1$ then

$$\underline{b_2}(E(n)_{pq}) = b_2(E(n)) = 12n - 2$$
$$\underline{b_2^+}(E(n)_{pq}) = b_2^+(E(n)) = 2n - 1$$
$$\underline{\sigma}(E(n)_{pq}) = \sigma(E(n)) = -8n$$

$E(n)_{pq}$ is spin ($\Leftrightarrow Q$ even) iff $n \equiv 0 \pmod 2$
$\wedge\ pq \equiv 1 \pmod 2$

cor: $E(n)_{pq} \overset{C^0}{\simeq} E(m)_{rs}$ iff $n = m \wedge (n$ odd or $pq \equiv rs \pmod 2))$

(Freedman)

thm: $\mathrm{Basic}_{SW}(E(n)_{pq}) = \{ PD(k\, \overset{\text{multiple fibres}}{\downarrow} f_{pq}) \mid k \equiv npq - p - q\ (2),\ |k| \leq npq - p - q \}$

if $\boxed{n \geq 2}$

cor: $E(n)_{pq} \overset{C^\infty}{\simeq} E(m)_{rs}$ iff $\{r, s\} = \{p, q\}$.

$E(1)_{pq}$ is more difficult since $b_2^+(E(1)_{pq}) = 1$

thm: For $n = 1$ the diffeomorphism classes are :

· $E(1)_{1p} \simeq E(1)_{pq}$ for $p, q \geq 1$ (N.B. $E(1) = E(1)_{11}$)

· $E(1)_{pq} \simeq E(1)_{rs}$ iff $\{p, q\} = \{r, s\}$ for $p, q, r, s > 1$

(Friedman)

3-k

3.4 Classification

Let X be a complex manifold and K_X its canonical bundle

A basis $f_0, f_1, \dots, f_m$ for $H^0(K_X^{\otimes n})$ induces a map

$$\phi_n : X \dashrightarrow \mathbb{C}P^m \qquad \text{(meromorphic/holomorphic)}$$
$$x \longmapsto [f_0(x) : f_1(x) : \cdots : f_m(x)]$$

called the pluricanonical map.

dfn: The Kodaira dimension of X is

$$\kappa(X) = \sup\{\dim \operatorname{im}(\phi_n) \mid n \geq 1 \text{ s.t. } H^0(K_X^{\otimes n}) \neq 0\}$$

Rk:
- $\kappa(X) \in \{0, 1, \dots, \dim X\}$ or $\kappa(X) = -\infty$
- $\kappa(X) = -\infty$ iff $\dim H^0(K_X^{\otimes n}) = 0 \ \forall n \geq 1$
- κ is additive : $\kappa(X \times Y) = \kappa(X) + \kappa(Y)$.
- If $\kappa(X) \geq 0$ then $\kappa(X) = \min\{k \geq 0 \mid \underset{\uparrow \dim H^0(K^{\otimes n})}{P_n(X)} / n^k \text{ is bounded}\}$

prp: κ is a birational invariant, hence preserved by blow-u

e.g.
- $\kappa(\mathbb{C}P^n) = -\infty$
- $\kappa(C) = -\infty$ if $g = 0$, 0 if $g = 1$ and 1 if $g \geq 1$.

Now let's look at surfaces:

$K = -\infty$:

e.g. $\mathbb{C}P^2$
e.g. $\mathbb{C}P^1 \times C$

e.g. ruled surfaces.

dfn: S is ruled if it admits a ~~map~~ holomorphic
$$\pi : S \to C$$
to some curve C s.t. $\pi^{-1}(t) \overset{hol}{\cong} \mathbb{C}P^1$ for every $t \in C$.

$K(S) = -\infty$ because $K_F \cong K_S|_F \otimes \nu_F$ for $F = \pi^{-1}(t)$
and ν_F is trivial.
If $K_S^{\otimes n}$ has sections, so does $K_F^{\otimes n}$.

thm: If S is ruled, minimal and $\pi_1(S) = 0$ then
$S \overset{hol}{\cong} F_n$ for some $n \geq 0$ or $S \overset{hol}{\cong} \mathbb{C}P^2$.

dfn: $F_n = \mathbb{P}(L_n \oplus \mathbb{C}) \overset{\pi}{\to} ~~d~~ \mathbb{C}P^1$ for L_n s.t. $c_1(L_n) = n$
is the n-th Hirzebruch surface.

thm:
- $F_n \overset{hol}{\cong} F_m$ iff $n = m$
- $F_n \overset{C^\infty}{\cong} \begin{cases} \mathbb{C}P^1 \times \mathbb{C}P^1 & \text{if } n \equiv 0 \pmod 2 \\ \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$ $\leftarrow$ = if $n = 0, 1$
- F_n is minimal iff $n \neq 1$

cor: $K(S) = -\infty \wedge \pi_1(S) \wedge S$ minimal $\Longrightarrow S \overset{C^\infty}{\cong} \mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

3 m

$\kappa = 0$:

thm: If S is minimal, $\kappa(S) = 0$ and $\pi_1(S) = 0$ then S is a $K3$ surface

cor: Then $S \overset{C^\infty}{\simeq} E(2)$.

$\kappa = 1$:

thm: If $\kappa(S) = 1$ and S is minimal, then S is elliptic.

Rk: The converse is not true: $\kappa(E(1)) = -\infty$ and $\kappa(E(2)) = 0$.

thm: If moreover $\pi_1(S) = 0$ then $S \overset{C^\infty}{\simeq} E(n)_{p,q}$
for some ~~p~~ $n, p, q \in \mathbb{N}$ s.t. $\gcd(p,q) = 1$.

(Recall: $E(n)_{pq} \overset{C^\infty}{\simeq} E(m)_{pq}$ iff $n = m \wedge \{p,q\} = \{r,s\}$
or $n = m = 1 \wedge 1 \in \{p,q\} \cap \{r,s\}$.)

$\kappa = 2$:

dfn: If $\kappa(S) = 2$ then S is of general type

e.g. $C_1 \times C_2$ ~~for~~ if $g(C_i) \geq 2$.

Not much is known about the classification of these.

3-n

The study of the homeomorphism types of complex surfaces of general type is called the "geography question"

thm: For any $b_1, b_2 \in \mathbb{N}_0$, there is a finite number of non-diffeomorphic complex surfaces of general type with these Betti numbers

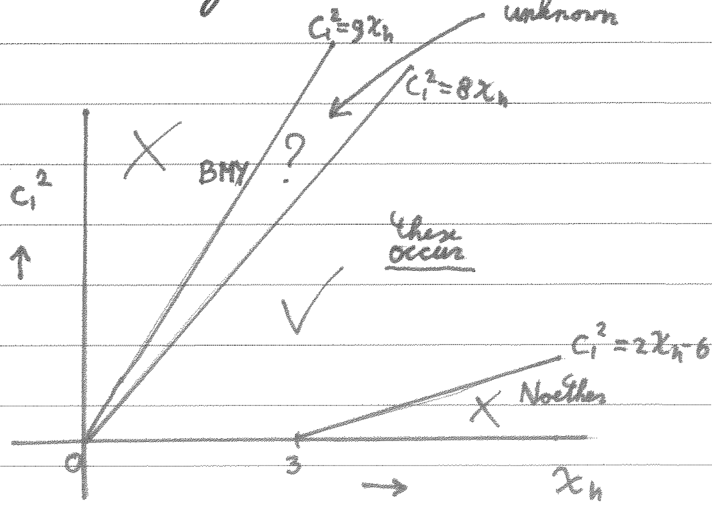
Let $\chi_h := \sum_{i=0}^{2} \dim H^i(S, \mathcal{O}_S)$, then $\chi_h = \frac{1}{4}(\sigma + \chi) \in \mathbb{Z}$
Moreover, $c_1^2 = 3\sigma + 2\chi$.

thm: If S is minimal and of general type then
- $c_1^2(S) > 0$
- $c_2(S) > 0$
- $2\chi_h(S) - 6 \underset{\uparrow}{\leq} c_1^2(S) \underset{\uparrow}{\leq} 9\chi_h(S)$

Noether inequality   Bogomolov-Miyaoka-Yau inequality

thm: Any $c_1, \chi_h \in \mathbb{Z}$ s.t. these inequalities hold and $c_1^2 \leq 8\chi_h$ is realised by a minimal surface of general type.

Rk: Requiring $\pi_1(S) = 0$ introduces holes.



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Tentamen: Datum: Uitslag:

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In summary

thm: If S is a minimal complex surface with $\pi_1(S)=0$, then S is diffeomorphic to one of the following:

- $\mathbb{C}P^2$
- $\mathbb{C}P^1 \times \mathbb{C}P^1$
- $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$
- $E(n)_{pq}$ for $n,p,q \in \mathbb{N}$ s.t. $\gcd(p,q)=1$
- a surface of general type

Rk: This includes the K3 surface $K3 \overset{C^\infty}{\cong} E(2)$

Rk: All instances of "diffeomorphic" in this section (3.4) can be replaced by "deformation equivalent".

dfn: S_1 and S_2 are deformation equivalent if there exists a surjective proper holomorphic map

$$\pi : \mathcal{T} \longrightarrow G$$

between connected complex spaces with smooth fibres and $t_1, t_2 \in G$ s.t. $S_i \overset{hol}{\cong} \pi^{-1}(t_i)$.
(A complex space is a space modelled on analytic subsets of $\mathbb{C}^n$)

conj: For simply connected complex surfaces deformation equivalence and diffeomorphisms define the same relation (True for $\kappa \leq 1$)

Rk: The full Enriques-Kodaira classification includes several classes of complex surfaces that are not simply connected