MORSE THEORY

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1. **Lecture One. Propaganda.** Poincaré conjecture in dimensions \( \geq 5 \).

1.1. **Singularities of smooth maps.** Let \( M, M' \) be smooth manifolds\(^1\), and \( f : M \to M' \) a smooth map. We define
\[
\text{Crit}(f) := \{ x \in M : \text{rank}_x f < \min(\dim M, \dim M') \}
\]
the set of **critical points** of \( f \), and \( f\text{Crit}(f) \subset M' \) the set of **critical values** of \( f \). \( M\setminus\text{Crit}(f) \) and \( \mathbb{R}\setminus f\text{Crit}(f) \) are then said to consist of **regular points** and **regular values**, respectively.

Note that \( \text{Crit}(f) \subset M \) is closed.

1.1.1. **Abundance of regular values: Sard’s theorem.** Recall that a subspace \( X \subset \mathbb{R}^m \) is said to have **measure zero** if, for all \( \varepsilon > 0 \) there is a sequence of balls \( (B_n)_{n \geq 0} \), with
\[
\sum_n \text{vol}(B_n) < \varepsilon, \quad \bigcup_n B_n \supset X.
\]
One immediately checks that:

- \( (X_n)_{n \geq 0} \) have measure zero \( \implies \bigcup_n X_n \) has measure zero;
- \( g : \mathbb{R}^m \to \mathbb{R}^m \) smooth, \( X \subset \mathbb{R}^m \) of measure zero \( \implies g(X) \subset \mathbb{R}^m \) has measure zero.

Hence the following notion is well-defined: a subspace \( X \subset M \) is said to have measure zero if there exists a smooth atlas \( A = \left\{ (U_i, \varphi_i) \right\} \) of \( M \), with each \( \varphi_i(X \cap U_i) \subset \mathbb{R}^m \) of measure zero.

Recall now:

**Theorem 1 (Sard).** If \( f : M \to M' \) is smooth, \( f\text{Crit}(f) \subset M' \) has measure zero.

**Proof.** See [9] or [29]. \( \square \)

Hence 'almost all' values are regular.

When \( \dim M \ll \dim M' \), there is a 'lot of space' to deform \( f \) inside \( M' \), and we can always remove the singularities of \( f \) – i.e., perturb it slightly to a \( f' \) with \( \text{Crit}(f') = \emptyset \).

**Theorem 2 (Whitney).** Every \( f : M \to M' \) is \( C^\infty \)-close to an injective immersion if \( \dim M' \geq 2 \dim M \), and every \( M^m \) embeds in \( \mathbb{R}^{2m} \).

In the other extreme, if \( M \) is compact, without boundary, \( \text{Crit}(f) \neq \emptyset \).

So we cannot get rid of singularities of functions.

1.1.2. **Singularities of Functions.** We consider the assignement \( M \mapsto C^\infty(M) \).

**Meta-principle 1 : Min-Max:** "The more complicated the topology of \( M \), the greater the number of critical points of functions on it."

**Example 1.** If \( f : T^n \to \mathbb{R} \), then \( |\text{Crit}(f)| \geq n + 1 \).

For more, see Min-Max theory \( \circledast \).

**Meta-principle 2 : Morse-Smale:** "The dynamics of a nice enough \( f \in C^\infty(M) \) reconstructs \( M \) smoothly."

**Example 2.** Suppose \( M^m \) is compact without boundary, and \( f : M \to \mathbb{R} \) has exactly two critical points. Then \( M^m \) is homeomorphic to \( S^m \).

\(^1\)Throughout these notes, by a manifold, we mean a Hausdorff, second-countable topological space, equipped with a maximal smooth atlas.
Aside: Poincaré Conjecture & Homotopy Spheres

Remark 1. It is not claimed that $M^m$ is diffeomorphic to $S^n$, with its standard smooth structure. In fact, in [14], Milnor constructs smooth $S^3$-bundles $p : M \to S^4$, for which there cannot exist $B^8$ with

$$\partial B = M, \quad H^4(B; \mathbb{Z}) = 0,$$

and carries a smooth $f \in C^\infty(M)$ with exactly two non-degenerate critical points. This implies that $M$ is homeomorphic to $S^4$, but not diffeomorphic to it; such manifolds are called exotic spheres.

Definition 1. A homotopy sphere is a smooth, oriented manifold $M^m$, homotopy-equivalent to $S^n$.

Note that if $M^m$ is a homotopy sphere, then $\pi_1(M) = \{1\}$, and $\pi_*(M; \mathbb{Z}) \cong \pi_*(S^n; \mathbb{Z})$. Conversely, if $M^m$ is simply connected and $\pi_*(M; \mathbb{Z}) \cong \pi_*(S^n; \mathbb{Z})$, then $M^m$ is a homotopy sphere; indeed, in that case $\pi_*(M) \cong \pi_*(S^n)$ by Hurewicz’ theorem. Now, a generator $[\alpha] \in \pi_m(S^n)$, $\alpha : S^n \to M$, gives rise to a homotopy equivalence $S^n \xrightarrow{\sim} M$.

Homotopy spheres are the object of the famous

Theorem 3 (Poincaré Conjecture). A homotopy sphere $M^m$ is homeomorphic to $S^m$.

Observe that the smooth version of the theorem, claiming that homotopy spheres are diffeomorphic to $S^m$, is decidedly false in light of the existence of exotic spheres.

Example 3. Milnor’s stand-up torus... ☝️. How does one make a drawing ?

• Present $(\mathbb{T}^2, f)$ and its 4 critical points;
• 'Non-degenerate' allows normal forms around the points; describe them;
• Show how the topology of $f^{-1}(-\infty, t]$ changes as $t$ varies. ('reconstruction').

Note that :

1. If $[a, b]$ contains no critical values of $f$, then the diffeomorphism type of $f^{-1}(-\infty, t]$ is independent of $t \in [a, b]$;
2. When $t$ crosses a critical value $t = c$, we have

$$f^{-1}(-\infty, c + \varepsilon] = f^{-1}(-\infty, c - \varepsilon] \cup H_\lambda,$$

where $H_\lambda$ denotes a "handle" $H_\lambda \sim \mathbb{D}^\lambda \times \mathbb{D}^{m-\lambda}$.

In view of Sard’s theorem, (1) suggests that we subdivide our task of understanding the topology of $M$.

Definition 2. A cobordism $\mathcal{C} = (W; M_0, f_0, M_1, f_1)$ from $M_0^m$ to $M_1^m$ is a smooth manifold $W^{m+1}$, together with a decomposition of its boundary as $\partial W = \partial_0 W \bigsqcup \partial_1 W$, together with diffeomorphisms $f_i : \partial_i W \xrightarrow{\sim} M_i$.

If $M_i$ are oriented (as will usually be the case), we assume further that $W$ is oriented, and that $f_0$ be orientation-preserving, while $f_1$ is orientation-reversing; we refer to $\mathcal{C}$ as an oriented cobordism between $M_0$ and $M_1$. We will also refer to $\partial_0 W$ as the incoming boundary of $W$, and to $\partial_1 W$ as the outgoing boundary of $W$. 
We regard a cobordism $C$ as a 'morphism' $M_0 \rightsquigarrow M_1$ of sorts. The maps will be suppressed from the notation when no confusion can arise.

**Definition 3.** Two cobordisms $C = (W, M_0, f_0, M_1, f_1)$ and $\tilde{C} = (W', M'_0, f'_0, M'_1, f'_1)$ are said to be **equivalent over** $M_0$ if there exists an oriented diffeomorphism $F : W \xrightarrow{\sim} W'$, $f'_0 \circ F = f_0$.

A cobordism $C = (W, M_0, f_0, M_1, f_1)$ is called:

- **trivial** if it is equivalent over $M_0$ to $(M_0 \times [0, 1]; M_0, M_1)$;
- **an h-cobordism** if $\partial_0 W \leftrightarrow W$ are homotopy equivalences.\(^2\)

One of the goals of this course is to prove the following fundamental

**Theorem 4 (Smale’s h-cobordism theorem).** If $\pi_1 M_0 = \{1\}$ and $\dim M \geq 5$, any h-cobordism over $M_0$ is trivial.

**Corollary 1 (Characterization of disks).** If $M^m$ is a contractible, smooth, compact manifold, and $\pi_1(\partial M) = \{1\}$, then $M \simeq D^m$ if $m \geq 6$.

**Proof.** Choose an embedding $j : D^m \hookrightarrow M^m$, and let

$$\tilde{M} := M \setminus j(D^m).$$

Then $j : \partial D^m \hookrightarrow \tilde{M}$ induces a long exact sequence

$$\cdots \rightarrow H_{k+1}(\tilde{M}, j(\partial D^m); \mathbb{Z}) \rightarrow H_k(j(\partial D^m); \mathbb{Z}) \rightarrow H_k(\tilde{M}; \mathbb{Z}) \rightarrow H_k(\tilde{M}, j(\partial D^m); \mathbb{Z}) \rightarrow \cdots$$

But note that $M \simeq \tilde{M} / j(\partial D^m)$, so

$$H_\ast(\tilde{M}, j(\partial D^m); \mathbb{Z}) = H_\ast(\tilde{M} / j(\partial D^m); \mathbb{Z}) = H_\ast(M; \mathbb{Z}).$$

By hypothesis, $H_\ast(M; \mathbb{Z}) = 0$, so we conclude that $H_\ast(j_\partial) : H_\ast(j(\partial D^m); \mathbb{Z}) \rightarrow H_\ast(\tilde{M}; \mathbb{Z})$ is an isomorphism. Hence by Whitehead’s theorem, and the fact that $\pi_1(\tilde{M}) = \{1\}$, we see that $j_\partial : \partial D^m \hookrightarrow \tilde{M}$ is a homotopy equivalence. Thus $(\tilde{M}; \partial D^m, \partial M)$ is an h-cobordism.

By Smale’s theorem, there is an equivalence

$$F : \tilde{M} \xrightarrow{\sim} \partial D^m \times [0, 1]; \partial D^m \times \{0\}, \partial D^m \times \{1\}).$$

But $M$ is clearly recovered as

$$\partial D^m \xrightarrow{j_\partial} \tilde{M} \xrightarrow{1} D^m \xrightarrow{1} M$$

so $M \simeq (S^{m-1} \times [0, 1]) \cup_{S^{m-1}} D^m \simeq D^m$. \(\square\)

**Corollary 2 (Poincaré conjecture in high dimensions).** $M^m$ homotopy sphere, $m \geq 6 \implies M$ is homeomorphic to $S^m$.

**Proof.** As before, start with an embedding $j : D^m \hookrightarrow M^m$, and let $\tilde{M} := M \setminus j(D^m)$, so that $H_k(\tilde{M}, j(D^m); \mathbb{Z}) = H_k(M; \mathbb{Z})$ still holds. The long exact sequence of the pair $(\tilde{M}, j(\partial D^m))$ implies that

$$H_k(j(D^m); \mathbb{Z}) = H_k(\tilde{M}; \mathbb{Z}), \quad k \leq n - 2,$$

since $H_k(M; \mathbb{Z}) = H_k(S^m; \mathbb{Z}) = 0$ for $k \neq 0, m$. For the case $k = m - 1$ we have

$$0 \rightarrow H_m(M; \mathbb{Z}) \rightarrow H_{m-1}(j(D^m); \mathbb{Z}) \rightarrow H_{m-1}(\tilde{M}; \mathbb{Z}) \rightarrow 0;$$

\(^2\)Note that this is the homotopy-theoretic version of (1) above.
but note that maps the fundamental class of $M$ to that of $j(\partial \mathbb{D}^m)$:

$$H_m(M; \mathbb{Z}) \ni [M] \mapsto [j(\partial \mathbb{D}^m)] \in H_{m-1}(j(\partial \mathbb{D}^m); \mathbb{Z}),$$

and thus $H_{m-1}(\tilde{M}; \mathbb{Z}) = 0$.

Hence $\pi_1(\partial M) = \{1\}$, $H_\bullet(M; \mathbb{Z}) = 0$, so by Corollary 1, there is a diffeomorphism $i : \mathbb{D}^m \sim \rightarrow \tilde{M}$.

Consider the diffeomorphism $f := j \circ i^{-1} : i(\partial \mathbb{D}^m) \sim \rightarrow j(\partial \mathbb{D}^m)$. In general, it is not possible to extend $f$ to a diffeomorphism $\tilde{f} : i(\mathbb{D}^m) \sim \rightarrow j(\mathbb{D}^m)$; however, we can extend $f$ to a homeomorphism $\tilde{f} : i(\mathbb{D}^m) \sim \rightarrow j(\mathbb{D}^m)$ by the so-called 'Alexander trick':

$$\tilde{f} : i(x) \mapsto |x| f\left(\frac{x}{|x|}\right).$$

We can now define a homeomorphism $F : \mathbb{S}^m \rightarrow M^m$ by

$$\begin{array}{c}
\mathbb{D}^m_- \xrightarrow{f_{\mathbb{D}^m_-}} \mathbb{S}^m \xrightarrow{F} \mathbb{D}^m_+ \\
F^{-1} \circ j \downarrow \downarrow \downarrow
\end{array}$$

1.2. Exercises.

(1) Recall the definition of the weak and strong topologies in the function spaces $C^k(M, M')$, and that $C^r_W(M, M')$ has a complete metric.

(2) Show that $\text{Prop}(M, M') \subset C^0_S(M, M')$ is a connected component.

(3) Let $U \subset M$ be open. The restriction map $C^r(M, M') \rightarrow C^r(U, M')$, $0 \leq r \leq \infty$, is continuous for the weak topology, but not always the strong.

However, it is an open map for the strong topologies, and not always for the weak topology.

(4) A submanifold $X \subset W$ of a manifold with boundary is called neat if $\partial X = X \cap \partial W$, and $X$ is not tangent to $\partial W$ at any point $x \in \partial X$. Show that if $y \in M$ is a regular value for $f : W \rightarrow M$ and $f|\partial W : \partial W \rightarrow M$, then $f^{-1}(y) \subset W$ is a neat submanifold.

(5) If $f : M \rightarrow M'$ is smooth, and $X \subset M'$ is a submanifold, we say that $f$ is transverse to $X$, written $f \pitchfork X$, if

$$\text{im} \, df + T_{f(x)}X = T_{f(x)}M', \quad x \in f^{-1}(X).$$

Suppose now that $W$ is a manifold with boundary, and $f : W \rightarrow M'$ is smooth. If $f, f|\partial W \pitchfork X$, then $f^{-1}X \subset W$ is a neat submanifold, and codim($f^{-1}X \subset W$) = codim($X \subset M'$).

(6) Every closed subspace $X \subset M$ can be described as $X = f^{-1}(0)$, where $f : M \rightarrow \mathbb{R}$ is a smooth function.

(7) Can you find a smooth $f : \mathbb{T}^2 \rightarrow \mathbb{R}$ with exactly three critical points?

(8) Show that if $W$ if a compact manifold with boundary, there can be no continuous map $r : W \rightarrow \partial W$ extending $\text{id}_{\partial W}$. 

A general goal of this course is to understand how to extract information about the topology of $M$ by means of 'good' functions $f : M \to \mathbb{R}$.

We will be mostly concerned with compact manifolds without boundary, but many natural constructions lead us away from this more manageable case. When $M$ is non-compact, we will typically demand that $f : M \to \mathbb{R}$ be proper.

Aside: proper maps

Recall that a map $f : M \to M'$ is said to be proper if $f^{-1}$ takes compact sets to compact sets. The subspace $\text{Prop}^k(M, M') \subset C^k(M, M')$ of proper maps is open in the strong $C^k$-topology, for every $k \geq 0$.

Exercise: if $f$ is proper, then $f(\text{Crit}(f)) \subset M'$ is closed.

One very strong reason to deal exclusively with proper maps is that non-proper maps may not reflect any of the topology of $M$. As an illustration, let us convene that an open manifold is a manifold, none of whose connected components is compact without boundary. Then we have

Theorem 5 (Gromov). On every open manifold $M$, there is $f \in C^\infty(M)$ with $\text{Crit}(f) = \emptyset$.

The catch is that such an $f$ cannot be proper.

Aside: Vector fields and their flows

Recall that, by the Fundamental Theorem of ODEs, a vector field $w \in \mathfrak{X}(M)$ defines a local flow. That is, there is

$$\phi : M \times \mathbb{R} \supset \text{dom}(\phi) \longrightarrow M,$$

where $\text{dom}(\phi)$ is an open containing $M \times \{0\}$, with the property that, for each $x \in M$, $c(t) := \phi^t(x)$ is the maximal trajectory of $w$ with initial condition $c(0) = x$.

Being a trajectory of $w$ means that $\frac{dc}{dt} = w \circ c$; by 'maximal trajectory' we mean that $c : \text{dom}(\phi) \cap \{x\} \times \mathbb{R} =: (a_x, b_x) \to M$ cannot be extended any further.

Note that $\phi^t(\phi^s(x)) = \phi^{t+s}(x)$ whenever either side of the equation is defined.

When $\text{dom}(\phi) = M \times \mathbb{R}$, we say that $\phi$ is the flow of $w$; in this case, $\phi$ determines a group homomorphism $\phi : (\mathbb{R}, +) \to (\text{Diff}(M), \circ)$, and we will say that $w$ is complete. Exercise: $w$ is complete if it is compactly supported.

However,
Example 4. Neither ∂/∂t ∈ 𝕀(ℝ ∖ 0) nor (1 + t^2)∂/∂t ∈ 𝕀(ℝ) are complete\(^3\).

There is a classical condition to be imposed on w to ensure that it give rise to a flow.

Definition 4. A Riemannian metric g on a manifold M is called complete if the geodesics of its Levi-Civita connection are defined at all times.

Complete Riemannian metrics exist on all manifolds of finite dimension.

Definition 5. A vector field w ∈ 𝕀(ℳ) is said to have bounded velocity if there exists a complete Riemannian metric g on M, for which ∥w∥ is bounded by some real number K :

\[ \sup_{x ∈ ℳ} ∥w(x)∥ ≤ K < +∞. \]

Lemma 1. Let (ℳ, g) be complete.

(1) (a, b) ⊂ ℝ a bounded interval, and c : (a, b) → M a curve of finite length :

\[ \int_a^b ∥c'(t)∥ dt < ∞. \]

Then imc ⊂ M is precompact.

(2) Suppose c(t) is a maximal trajectory of w ∈ 𝕀(ℳ), c : J → M, where J ⊂ ℝ is an interval containing 0. Then :

- \( [0, +∞) ∉ J \implies \int_0^∞ ∥c'(t)∥ dt = ∞; \)
- \( (−∞, 0) ∉ J \implies \int_0^- ∥c'(t)∥ dt = ∞; \)

Proof. (1) : It suffices\(^4\) to show that, for every ε > 0, there exist \( x_0, ..., x_N \) ∈ Cl im c such that the ε-balls around \( x_i \) cover it : \( \bigcup_i^N B_ε(x_i) \supset Cl im \ c. \) But

\[ \int_a^b ∥c'(t)∥ dt < ∞ \implies ∃ a = t_0 < t_1 < ⋯ < t_N = b, \quad \int_{t_i}^{t_{i+1}} ∥c'(t)∥ dt < ε, \]

so

\[ Cl(im \ c) \subset \bigcup_{0}^N B_ε(c(t_i)). \]

(2) : If c is maximal, and \( [0, +∞) ∉ J \), then c(t) has no limit point as \( t → b, \)

\( b := sup \{ t : t ∈ J \} < ∞. \) It then follows from the first part of the lemma that

\[ \int_0^b ∥c'(t)∥ dt = ∞. \]

The other case is completely analogous. □

Definition 6. An isotopy ψ of a smooth manifold M is a smooth map

\[ ψ : M × J → M, \]

where J ⊂ ℝ is an interval containing 0, each \( ψ_t := ψ(\cdot, t) : M → M \) is a diffeomorphism, and \( ψ_0 = id_M. \)

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\(^3\)For the second example, note that a solution curve c(t) to \( (1 + t^2)∂/∂t \) with initial condition c(0) = 0 is c(t) = tan t, which cannot be extended beyond \( (-π/2, +π/2) \).

\(^4\)A metric space \( (X, d) \) is called totally bounded if, for every ε > 0, X can be covered by finitely many ε-balls. A complete metric space is compact if it is totally bounded. Indeed, it is clear that any compact space is totally bounded. On the other hand, if a space is totally bounded, to show that it is compact it is enough to show that every sequence \( (x_n)_{n≥0} \) has a Cauchy subsequence \( (x_{n_k})_{k≥0} \). Cover X with finitely many balls \( B_1, ..., B_N \) of radius 1; then one of the balls, say \( B_1 \), must contain infinitely many terms of \( (x_n) \). This defines a subsequence \( s_1 ⊂ (x_n) \), and the distance between any two points in \( s_1 \) is no greater than 1. Now cover \( B_1 \) by finitely many balls of radius 1/2; again, we can select a subsequence \( s_2 \) of \( s_1 \) of points lying in one single 1/2-ball. Inductively, we define then a sequence of subsequences \( (x_n) ⊃ ·⋯⊂ s_k ⊃ s_{k+1} ⊃ ·⋯ \), with each \( s_k \) lying in a ball of radius 1/k; hence a sequence \( x_{n_k} ∈ s_k \setminus s_{k+1} \) must be Cauchy.
The flow of a complete vector field is an example of an isotopy.

An isotopy \( \psi \) gives rise to a **time-dependent vector field**, i.e., a one-parameter family \( t \mapsto w_t \) of vector fields on \( M \), defined by

\[
\frac{d\psi_t}{dt} = w_t \circ \psi_t, \quad t \in J.
\]

**Remark 2.** A time-dependent vector field \( w_t \) can of course be regarded as an autonomous vector field \( \tilde{w} \) on \( M \times J \), to which the above discussion applies to produce a local flow

\[
\tilde{\phi} : (M \times J) \times \mathbb{R} \supset \text{dom}(\tilde{\phi}) \rightarrow M \times J.
\]

Note however that \( \tilde{\phi} \) gives rise to an isotopy of \( M \times J \), not of \( M \) alone. This can be remedied as follows: consider the autonomous \( \hat{w} := w_t + \partial/\partial t \in X(M) \times J) \).

Let \( w_t \) be complete, and let us denote its flow by \( \hat{\phi} \).

\[
\hat{\phi}_s(x,t) \in M \times \{t + s\},
\]

so \( \hat{\phi}_{s+t}(x,t) = \hat{\phi}_t \circ \hat{\phi}_s(x,t) \) wherever this makes sense; in particular, \( s \mapsto \text{pr}_M \circ \hat{\phi}_s \) gives rise to an isotopy of \( M \).

We conclude that:

**Lemma 2.** Time-dependent vector fields of bounded velocity give rise to isotopies.

**Proof.**

We conclude this aside by recalling a very useful formula from Calculus. Suppose \( \psi \) is an isotopy of \( M \) with corresponding time-dependent vector field \( w_t \in X(M) \).

Suppose \( t \mapsto \eta_t \) is a time-dependent section of some tensor bundle \( E := \bigwedge^p TM \otimes (\bigwedge^q T^* M) \).

**Lemma 3.** \( \frac{d}{dt}(\psi_t^* \eta_t) = \psi_t^* \left( L(w_t)\eta_t + \frac{d\eta_t}{dt} \right) \).

**Exercise:** Prove the Lemma.

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**Proof of Structure Theorem I.** Suppose

\[
f \text{Crit}(f) \cap [a, b] = \emptyset.
\]

Then \( f \text{Crit}(f) \cap [a - \varepsilon, b + \varepsilon] = \emptyset \) for small enough \( \varepsilon > 0 \). Choose

\[
g : [a - \varepsilon, b + \varepsilon] \rightarrow [0, 1]
\]

such that

\[
g(t) = \begin{cases} 1 & \text{if } t \in [a - \varepsilon/3, b + \varepsilon/3]; \\ 0 & \text{if } t \notin [a - 2\varepsilon/3, b + 2\varepsilon/3]. \end{cases}
\]

Let

\[
w := \frac{-(g \circ f)}{||\nabla f||^2} \nabla f \in X(M),
\]

where \( || \cdot || \) refers to some auxiliary (complete) Riemannian metric \( g \) on \( M \) and \( \nabla f \) denotes the vector field defined by \( g(\nabla f, v) = df(v) \).

Observe that

\[
(L(w)f)(x) = \begin{cases} -1 & \text{if } f(x) \in [a - \varepsilon/3, b + \varepsilon/3]; \\ 0 & \text{if } f(x) \notin [a - 2\varepsilon/3, b + 2\varepsilon/3]. \end{cases}
\]
$f$ being proper, $w$ is compactly supported, and so gives rise to a flow

$$\phi : M_b \times \mathbb{R} \to M_b, \quad \phi_t(M_b) \subset M_{b-t}. $$

In particular, we have a diffeomorphism

$$\phi_{b-a} : M_b \cong M_a,$$

and

$$\phi|_{M_b \times [0,b-a]} : M_b \times [0,b-a] \to M_b$$

is a strong deformation retraction of $M_b$ onto $M_a$. □

This idea that 'in the absence of critical points we can push down $M_t$' can be turned around to detect critical points of a $f \in C^\infty(M)$.

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**Aside: Palais-Smale Condition C**

Fix a complete Riemannian manifold $(M,g)$, and let $f : M \to \mathbb{R}$ be given.

**Definition 7.** We say that $f$ satisfies **Condition C** if, whenever a sequence $(x_n)_{n \geq 0}$ in $M$ is such that

- $(|f(x_n)|)_{n \geq 0} \subset \mathbb{R}$ is bounded, and
- $\|\nabla f(x_n)\| \to 0$ as $n \to \infty$,

then there is a subsequence $(x_{n_k})_{k \geq 0}$ converging in $M$.

Observe that any proper $f$ satisfies Condition $C$ automatically.

---

**Lemma 4.** Suppose $f$ is bounded below, and $f$ satisfies Condition $C$. Then the flow $\phi^t$ of $-\nabla f$ is defined for all positive times, and for every $x \in M$, $\lim_{t \to +\infty} \phi^t(x)$ exists and is a critical point of $f$.

**Proof.** Let $B := \inf_{x \in M} f(x) > -\infty$, and consider the maximal trajectory

$$c(t) := \phi^t(x), \quad c : J \to M;$$

we wish to show that $[0, +\infty) \subset J$.

First define $F : (a,b) \to \mathbb{R}$ by $F(t) := f(c(t))$. Then

$$B \leq F(t) = F(0) + \int_0^t F'(s) ds = F(0) - \int_0^t \|\nabla f(c(s))\|^2 ds$$

$$\implies \int_0^t \|\nabla f(c(s))\|^2 ds \leq F(0) - B.$$

Since the RHS is independent of $t$, we conclude that

$$\int_0^b \|\nabla f(c(s))\|^2 ds \leq F(0) - B.$$

Let us argue by contradiction, and assume that $b$ were finite. By Schwarz’s inequality,

$$\int_0^b \|\nabla f(c(s))\| ds \leq \sqrt{\int_0^b ds \sqrt{\int_0^b \|\nabla f(c(s))\|^2 ds}} \leq \sqrt{b(F(0) - B)}.$$

This implies that $\int_0^b \|\nabla f(c(s))\| ds < +\infty$. But by Lemma 1, $b < +\infty$ implies that $\int_0^b \|\nabla f(c(s))\| ds$ is infinite; the contradiction shows that $b = +\infty$. 

But then
\[ \int_0^\infty \|\nabla f(c(s))\|^2 ds \leq F(0) - B \implies \|\nabla f(c(t))\|^2 \to 0, \text{ as } t \to \infty, \]
so \( \|\nabla f(c(t))\| \to 0. \) By Condition C, we can find \((t_n)_{n \geq 0}\) with \(c(t_n) \to x \in M; \) by continuity of \(df,\) we have
\[ x \in \text{Crit}(f). \]
\[ \square \]

We will return to this sort of argument in more detail when we deal with Min-Max theory.

2.1. Normal Forms. Having dealt with the regular case, we wish to understand the behavior of \(f\) around its singular points \(x \in \text{Crit}(f).\) Ideally, we should be able to provide a model for \(f\) around each critical point, depending only on the value of a (a priori known) finite number of derivatives of \(f\) at \(x.\)

For too badly behaved \(f,\) this is way too ambitious.

Example 5. The maps \(f_0, f_1 : \mathbb{R} \to \mathbb{R},\)
\[ f_0(t) = 0, \quad f_1(t) = \begin{cases} e^{-1/t} & \text{if } t \geq 0, \\ 0 & \text{if } t \leq 0. \end{cases} \]
both have \(0\) as a critical point, and their derivatives at \(0\) vanish to infinite order, and they behave quite differently at zero.

To weed out such behavior, and still hope to model the singularities of \(f,\) we should impose some non-degeneracy condition on the critical points \(x \in \text{Crit}(f).\)

Aside : Germs

Recall that if \(M, M'\) are smooth manifolds, and \(X \subset M\) is any subspace, we denote by
\[ C^\infty(M, M')_X = \{ [U, f] : X \subset U \subset M \text{ open}, f \in C^\infty(U, M') \} , \]
where \([U, f]\) denotes the germ of \(f\) along \(X:\)
\[ [U, f] = [U', f'] \iff \exists U'' \subset U \cap U', \ f|_{U''} = f'|_{U''}. \]
Two germs \([U, f], [U', f'] \in C^\infty(M, M')_X\) will be called equivalent if there exist \(U'' \subset U \cap U', V \supset f(X)\) opens, and embeddings \(j : U'' \hookrightarrow U\) and \(i : V \hookrightarrow M',\) with
\[ if|_{U''} = f'|_{U''}j. \]

An equivalence class of germs around \(X\) formalizes the notion of 'behavior' around \(X:\) two maps \(f, f' \in C^\infty(M, M')\) have the same behavior around \(X \subset M\) iff their germs along \(X\) are equivalent.

We will typically be lazy, and write \([f]\) (or just \(f\)) instead of \([U, f]\).

We will mostly be concerned with \(\mathcal{E} := C^\infty(\mathbb{R}^m, \mathbb{R})_0,\) the set of germs of real functions around zero. Note that this is a ring, with the operations
\[ [f] + [f'] := [f + f'], \quad [f] \cdot [f'] := [ff'], \]
with additive and multiplicative units \([0]\) and \([1]\) respectively. Observe that \(\mathcal{E}\) comes equipped with a natural surjective ring homomorphism
\[ \text{ev} : \mathcal{E} \to \mathbb{R}, [f] \mapsto f(0). \]
Since $\mathcal{E}/\mathbb{R}$ is a field, $\mathfrak{m} := \ker(\mathbb{e}v)$ is a maximal ideal in $\mathcal{E}$; observe that $[f] \notin \mathfrak{m}$ implies that $[f]^{-1} = [f^{-1}] \in \mathcal{E}$, so $\mathfrak{m} \triangleleft \mathcal{E}$ is the unique maximal ideal — that is, $\mathcal{E}$ is a local ring.

Observe that $[f] \in \mathfrak{m}$ iff

$$f(x) = \int_0^1 \frac{d}{dt} (f(tx)) dt = \sum_{i=1}^{m} \left( \int_0^1 \frac{\partial f}{\partial x_i}(tx) \right) \cdot x_i,$$

so $\mathfrak{m} = \sum_{i=1}^{m} \mathcal{E} \cdot x_i$; in particular, $m^2 = \sum_{i=1}^{m} \mathcal{E} \cdot x_i x_j$ and thus $[f] \in \mathfrak{m}^2$ iff $0 \in \text{Crit}(f)$. This implies that

$$m/m^2 \cong _{\mathbb{T}_0} \mathbb{R}^m,$$

$[f] + m^2 \mapsto d_0 f$, is an isomorphism of $\mathcal{E}$-modules.

This observation can be expanded by observing that $\mathbb{e}v$ extends to a ring homomorphism

$$\text{Tayl}: \mathcal{E} \to \mathbb{R}[[x_1, ..., x_m]], \quad [f] \mapsto \text{Tayl}(f) := \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^{\alpha} f}{\partial x^{\alpha}},$$

where for a multi-index $\alpha = (\alpha_1, ..., \alpha_m)$, $\alpha_i \geq 0$, we set

$$|\alpha| := \sum \alpha_i, \quad \alpha! := \prod_{i=1}^{m} \alpha_i!, \quad x^\alpha := \prod_{i=1}^{m} x_i^{\alpha_i}.$$  

The homogeneous part of degree $k$ of $\text{Tayl}(f)$, denoted by $\text{Tayl}^k(f)$, can be described in a slightly less coordinate-dependent fashion. Indeed, if $f: \mathbb{R}^m \to \mathbb{R}$ is a smooth map, then $df$ can be regarded as a smooth map $df : \mathbb{R}^m \to \text{Hom}(\mathbb{R}^m, \mathbb{R}) \cong \mathbb{R}^m$, and as such we can take $d(df) := d^2 f : \mathbb{R}^m \to \text{Hom}(\mathbb{R}^m, \text{Hom}(\mathbb{R}^m, \mathbb{R}))$. But recall from Calculus that $d^2 f$ lands inside $\text{Hom}^2(\mathbb{R}^m, \mathbb{R})$, i.e., $d^2 f(v, w)$ is symmetric in its arguments $v, w \in T_0 \mathbb{R}^m$. More generally, we denote by $d^k f$ the map $d(d^{k-1} f) : \mathbb{R}^m \to \text{Hom}^k(\mathbb{R}^m, \mathbb{R})$; in this notation,

$$\text{Tayl}^k(f) = \frac{1}{k!} d^k f.$$

**Lemma 5.** Let $[f] \in m^2$. Then

$$d^2 f(v, w) = [\tilde{v}, [\tilde{w}, f]](0) = [\tilde{w}, [\tilde{v}, f]](0),$$

where $\tilde{v}, \tilde{w}$ are any two germs of vector fields around zero extending $v, w \in T_0 \mathbb{R}^m$, respectively.

**Proof.** Note that

$$[\tilde{v}, [\tilde{w}, f]] - [\tilde{w}, [\tilde{v}, f]] = [[\tilde{v}, \tilde{w}], f](0) = m_0 f([\tilde{v}, \tilde{w}]) = 0$$

since $0 \in \text{Crit}(f)$. Hence $[\tilde{v}, [\tilde{w}, f]](0) = [\tilde{w}, [\tilde{v}, f]](0)$. But the LHS can be expressed as

$$[\tilde{v}, [\tilde{w}, f]](0) = d([\tilde{w}, f])(v),$$

which shows that it is independent of the choice of extension $\tilde{v}$, whereas

$$[\tilde{w}, [\tilde{v}, f]](0) = d([\tilde{v}, f])(w)$$

shows that it is independent of the extension $\tilde{w}$. Now express $f$ in coordinates and conclude that the quantity above equals $d^2 f(v, w)$ (exercise).

Now recall if $B: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is a symmetric bilinear form, there exist integers $0 \leq \lambda, \nu \leq m$ and a linear basis $(e_i)_{i=1}^m$ of $\mathbb{R}^m$ with

$$B(e_i, e_j) = \begin{cases} 
-1 & \text{if } i = j \text{ and } i \leq \lambda, \\
+1 & \text{if } i = j \text{ and } \lambda < i \leq m - \nu, \\
0 & \text{if } i \neq j \text{ or } i > m - \nu.
\end{cases}$$
The integer $\nu$ is called the **nullity** of $B$; the form is called **non-degenerate** if $\nu = 0$. The integer $\lambda$, on the other hand, is called the **index** of $B$. Observe that $\nu, \lambda, (m - \lambda - \nu)$ are the dimensions of the maximal subspaces $W \subset \mathbb{R}^m$ where $B$ restricts to zero, a negative-definite form, and a positive-definite form, respectively.

**Lemma 6.** Let $[f] \in m^2$, and let $\mbox{Jac} (f) \triangleleft E$ denote the ideal spanned by the partial derivatives $\frac{df}{dx_i}$. Then $d_0^2 f$ is non-degenerate only if $\mbox{Jac} (f) = m$.

**Proof.** Of course, $[f] \in m^2$ implies that $\mbox{Jac} (f) \subset m$, so one inclusion always holds.

Suppose $d_0^2 f$ were non-singular. Then $d_0 (df) = d_0^2 f : \mathbb{R}^m \to \text{Hom} (\mathbb{R}^m, \mathbb{R})$ is a linear isomorphism; hence by the Inverse Function Theorem, we can express the coordinates $x_i$ as

$$x_i = x_i (\partial f / \partial x_1, ..., \partial f / \partial x_m) \implies x_i = \sum_j a_{ij} \frac{\partial f}{\partial x_j},$$

for some $a_{ij} \in E$. Since the $x_i$’s span $m$, we have $\mbox{Jac} (f) = m$. □

Having described the local picture, we can transfer our definitions to the manifold setting:

**Definition 8.** The **Hessian** $\mbox{Hess}_x (f)$ is the bilinear form

$$T_x M \times T_x M \to \mathbb{R}, \quad \mbox{Hess}_x (f) (v, w) := d_0^2 f (v, w),$$

corresponding to the critical point $x \in \text{Crit} (f)$. A critical point $x \in \text{Crit} (f)$ is called **non-degenerate** if $\mbox{Hess}_x (f)$ is non-singular. If $x$ is a non-degenerate critical point, its **index** $\lambda = \lambda (f, x)$ is

$$\lambda (f, x) := \max \{ \dim W : \mbox{Hess}_x (f) |_W \text{ is negative-definite.} \}$$

A function $f \in C^\infty (M)$ will be called **Morse** if all of its critical points are non-degenerate.

We will write $\mbox{Morse} (M) \subset C^\infty (M)$ for the subspace of Morse functions.

**Lemma 7.**

1. $\mbox{Morse} (M)$ is open in the strong $C^2$-topology, $\mbox{Morse} (M) \subset C^2_0 (M)$.
2. $\lambda (f, x) + \lambda (-f, x) = \dim M$.

**Proof.** Immediate. □

We wish to prove now:

**Theorem 7** (Morse Lemma). If $f \in m^2 \setminus m^3$, there exists an embedding

$$\psi : 0 \in U \hookrightarrow \mathbb{R}^m, \quad \psi^* f = \frac{1}{2} \mbox{Hess}_x (f) \in E,$$

where we regard $\mbox{Hess}_x (f)$ as a smooth function $\mbox{Hess}_x (f) : T_x M \to \mathbb{R}$ by the rule $v \mapsto \mbox{Hess}_x (v, v)$.

We need a technical lemma first.

**Lemma 8** (Auxiliary Lemma). If $f \in m^2 \setminus m^3$, and $\delta \in m^2$, there exists a time-dependent vector field $w_t$ around zero, $t \in [0, 1]$, for which $[w_t, f + t\delta] = -\delta$ and $w_t (0) = 0$ for all $t$. 
Proof. Note that \( \delta \in \mathfrak{m}^3 \) implies that \( \nabla \delta \in \mathfrak{m}^2 \), so
\[
\nabla \delta = B(x)x, \quad B(0) = 0.
\]
On the other hand, \( \text{Jac}(f) = \mathfrak{m} \), so \( x = A(x)\nabla f \). Hence
\[
\begin{cases}
x = A(x) (\nabla (f + t\delta)) - tA(x) \nabla \delta \\
\nabla \delta = B(x)x
\end{cases} \implies (\text{id} + tA(x)B(x)) x = A(x) \nabla (f + t\delta).
\]
Now, \( B(0) = 0 \) ensures that
\[
x = C_t(x) \nabla (f + t\delta), \quad C_t(x) := (\text{id} + tA(x)B(x))^{-1} A(x),
\]
which means that each of germs of the coordinate functions \( x_i \) can be written as
\[
x_i = [v_i^t, f + t\delta]
\]
for some germ of time-dependent vector field \( v_i^t \).

Now write \( \delta = \sum_{i,j} a_{ij} x_i x_j \) and let
\[
w_t := \sum_{i,j} a_{ij} x_i v_i^t;
\]
then \( [w_t, f + t\delta] = -\delta \) as promised, and \( w_t(0) = 0 \) for all \( t \). \( \square \)

Proof of Theorem 7. First observe that
\[
f_t := (1 - t)f + \frac{t}{2} \text{Hess}(f) = f + t\delta, \quad \delta := \frac{1}{2} \text{Hess}(f) - f, \quad t \in [0, 1],
\]
defines a smooth family \( f_t \in \mathfrak{m}^2 \setminus \mathfrak{m}^3 \). Note that \( \delta \in \mathfrak{m}^3 \).

We seek a germ of isotopy \( \psi_t \) around 0, such that \( \psi_t(0) = 0 \) and
\[
\psi_t f_t = f, \quad t \in [0, 1]
\]
The latter condition is equivalent to
\[
0 = \frac{d}{dt} (\psi_t f_t) \iff L(w_t) f_t + \delta = 0,
\]
and the former to \( w_t(0) = 0 \), where \( w_t \) denotes the germ of time-dependent vector field corresponding to \( \psi_t \).

But by the Auxiliary Lemma 8, such \( w_t \) exists. \( \square \)

Definition 9. If \( f \in C^\infty(M) \) and \( p \in \text{Crit}(f) \) is non-degenerate, a Morse chart around \( p \) is an embedding \( \psi : U \hookrightarrow M \) of an open around 0 in \( \mathbb{R}^m \) putting \( f \) in normal form :
\[
\psi^* f = Q_{(f,p)},
\]
where \( Q_{(f,p)} \) stands for the standard quadratic form of index \( \lambda = \lambda(f,p) \), \( Q_{(f,p)} = -\sum_{1}^{\lambda} x_i^2 + \sum_{\lambda+1}^{m} x_i^2 \).

2.2. Exercises.

(1) If \( f \in \text{Morse}(M) \) and \( f' \in \text{Morse}(M') \) \( i = 0, 1, \) then \( F := \text{pr}_M^* f + \text{pr}_{M'}^* f' \in \text{Morse}(M \times M') \). Determine the critical points of \( F \) and their indices in terms of those of \( f, f' \).

(2) Give an example of isolated and non-isolated degenerate critical points.

(3) Show that if \( |f| \in \mathfrak{m} \setminus \mathfrak{m}^2 \), then \( f \) has the same behavior as \( d_x f \).

(4) Show that if \( f \in \text{Morse}(M^n) \) and \( |\text{Crit}(f)| = 2 \), then \( M \) is homeomorphic to \( S^n \).

(5) Show that every symmetric bilinear form \( B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is equivalent to (exactly) one of the form \( -\sum_{1}^{\lambda} x_i^2 + \sum_{\lambda+1}^{n} x_i^2 \), \( 0 \leq \lambda \leq n \).
3. Lecture Three. Abundance of Morse functions.

3.1. Thom Transversality Theorem. Recall that if $M, M'$ are smooth manifolds, we say that $f, f' \in C^\infty(M, M')$ have the same $k$-jet at $x \in M$ iff all the partial derivatives of $f$ and $f'$ at $x$ agree up to order $k$, in which case we write $j_k f(x) = j_k f'(x)$.

The collection

$$J_k(M, M') := \{ j_k f(x) : f \in C^\infty(U, M'), x \in U \}$$

of all $k$-jets of (partially defined) maps $M \to M'$ has a natural structure of smooth manifold. It comes equipped with source- and target maps,

$$s : J_k(M, M') \to M, \quad j_k f(x) \mapsto x$$
$$t : J_k(M, M') \to M', \quad j_k f(x) \mapsto f(x);$$

which are fibre bundles, and bundle maps

$$p_k : J_k(M, M') \to J_{k-1}(M, M'), \quad j_k f(x) \mapsto j_{k-1} f(x)$$

so that we have commuting diagrams

$$\begin{array}{ccc}
& J_k(M, M') & \\
M & \quad s & \quad t \\
& J_{k-1}(M, M') & \\
& p_k & \\
\end{array}$$

There is also an assignment

$$j_k : C^\infty(M, M') \to C^\infty(M, J_k(M, M')),$$

which we refer to as the $k$-jet map.

Recall that a subspace $A$ of a topological space $X$ is called residual if it is the countable intersection of open, dense subspaces:

$$A = \bigcap_{n \geq 0} U_n, \quad U_n \subset X \text{ open and } \text{Cl} U_n = X, \quad \forall n.$$ 

A topological space $X$ is called Baire if every residual subspace is dense.

**Theorem 8.** A residual subspace of a complete metric space is dense. Every weakly closed subspace of $C^r_\mathbb{R}(M, M')$ is a Baire space.

**Proof.** See [9].

We can now remind the reader of:

**Theorem 9** (Thom Transversality Theorem, v. 1). If $X \subset J_k(M, M')$ is a submanifold, then the space of $f \in C^r(M, M')$ with $j_k f \cap X$ is residual in $C^r_\mathbb{R}(M, M')$ for $r > k$, and is open if $X$ is closed.

Aside : Multijet bundles

We will make good use of an extension of Thom Transversality, whose setting we describe.
Fix an integer \( l > 0 \) and consider

\[
J^{(l)}_k(M, M') \subset \prod_{1 \leq i \leq l} J_k(M, M'), \quad J^{(l)}_k(M, M') := (\prod_{1 \leq i \leq l} s)^{-1} M^{(l)},
\]

where

\[
M^l \supset M^{(l)} := \{(x_1, ..., x_l): i \neq j \Rightarrow x_i \neq x_j\}.
\]

Then clearly \( J^{(l)}_k(M, M') \) is a bundle over \( M^{(l)} \), with projection

\[
s^{(l)} : (j_k f_1(x_1), ..., j_k f_l(x_l)) \mapsto (x_1, ..., x_l),
\]

and there is an induced multijet map

\[
j^{(l)}_k : C^k(M, M') \rightarrow \mathcal{C}^0(M^{(l)}, J^{(l)}_k(M, M'))
\]

\[
j^{(l)}_k f : M^{(l)} \ni (x_1, ..., x_l) \mapsto (j_k f_1(x_1), ..., j_k f_l(x_l)) \in J^{(l)}_k(M, M').
\]

**Theorem 10** (Thom Transversality Theorem, v. 2). If \( X \subset J^{(l)}_k(M, M') \) is a submanifold, then the space of \( f \in C^r(M, M') \) with \( j^{(l)}_k f \cap X \) is residual in \( C^r(M, M') \) for \( r > k \), and is open if \( X \) is closed.

**Proof.** See [7]. \( \square \)

Now we put these ideas to use.

**Definition 10.** The singularity set \( S_1 \subset J_1(M, \mathbb{R}) \) is the subspace defined by

\[
S_1 = \{j_1 f(x) : d_x f = 0\}.
\]

**Lemma 9.** \( S_1 \) is a closed submanifold, of codimension \( \text{codim} \, S_1 \subset J_1(M, \mathbb{R}) \) \( = \dim M \).

\[
x \in \text{Crit}(f) \iff j_1 f(x) \in S_1. \quad \text{Moreover, } x \text{ is non-degenerate iff } j_1 f \cap S_1 \text{ at } x.
\]

**Corollary 3.** \( \text{Morse}(M) \subset C^2(M, \mathbb{R}) \) is open and dense. If \( f \in \text{Morse}(M) \), \( \text{Crit}(f) \) is discrete.

**Proof.** Combine Lemma 9 with Theorem 9 for the first statement. For the second, observe that \( \text{codim} \, \text{Crit}(f) \subset M = \dim M \), so \( \text{Crit}(f) \) is a zero-dimensional submanifold. \( \square \)

**Definition 11.** A Morse function \( f \in \text{Morse}(M) \) is called **resonant** if there exist distinct critical points \( x, y \in \text{Crit}(f) \) at the same critical value : \( f(x) = f(y) \). Otherwise it is called **non-resonant**, and the space of all such will be written \( \text{Morse}_\neq(M) \).

**Lemma 10.** \( \text{Morse}_\neq(M) \subset C^2(M, \mathbb{R}) \) is open and dense.

**Proof.** First observe that \( \text{Morse}_\neq(M) \subset C^2(M, \mathbb{R}) \) is clearly open, so we need only show that it contains a dense subspace.

Consider the multijet bundle \( J^{(2)}_1(M, \mathbb{R}) \rightarrow M^{(2)} = M \times M \setminus \Delta_M \), and let \( S_\neq \subset J^{(2)}_1(M, \mathbb{R}) \) be the subspace defined by

\[
S_\neq := (S_1 \times S_1) \cap (t \times t)^{-1}(\Delta_\mathbb{R}).
\]

One readily sees that \( S_\neq \) is a submanifold of codimension \( 2 \dim M + 1 \), and hence \( j^{(2)}_1 f \cap S_\neq \) at \( (x_1, x_2) \) means \( j^{(2)}_1 f(x_1, x_2) \notin S_\neq \).

By Theorem 10, the subspace \( U \subset C^2(M, \mathbb{R}) \) of those \( f \) with \( j^{(2)}_1 f \cap S_\neq \) open and dense; thus \( \text{Morse}(M) \cap U \) is open and dense. But \( j^{(2)}_1 f \) maps \( (x_1, x_2) \) into \( S_\neq \) iff \( d_{x_1} f = 0 = d_{x_2} f \) and \( f(x_1) = f(x_2) \), so \( \text{Morse}_\neq(M) \supset \text{Morse}(M) \cap U \). \( \square \)
3.2. Concatenating and Factorizing Cobordisms.

In view of Lemma 10, any \( f \in C^\infty(M) \) can be perturbed ever so slightly to a non-resonant Morse function.

Suppose \( M \) is compact, so that \( \text{Crit}(f) \) is finite. Order the critical values

\[
\{c_1 < c_2 < \cdots < c_N\} = f(\text{Crit}(f)),
\]

and let \(-\infty = a_0 < a_1 < \cdots < a_{N-1} < a_N = +\infty\), with \( c_i \in (a_{i-1}, a_i) \) for every \( 1 \leq i \leq N \).

Then

\[
\begin{align*}
C_i & := (W_i, f^{-1}(a_{i-1}), f^{-1}(a_i)), & W_i := f^{-1}[a_{i-1}, a_i],
\end{align*}
\]

are cobordisms, and \( M = \bigcup_i W_i \). Note that \( f \equiv a_i \) for every \( 0 < i < N \), and \( f_i := f|_{W_i} \) contains a single critical point. We give this situation a special name:

**Definition 12.** A cobordism \( C = (W; M_0, M_1) \) is called elementary if there exists a smooth function \( f : W \to [a, b] \), with \( f \equiv a \) in a neighborhood of \( \partial W \), and in the positive coorientation if \( X \subset \partial W \), and in the positive coorientation if \( \partial X \subset \partial W \); here \( I(X, \varepsilon) = (\varepsilon, \varepsilon) \) if \( X \) is interior and \( I(X, \varepsilon) = [0, \varepsilon] \) if \( X \) lies in the boundary.

**Lemma 11** (Collars).

1. Collars exist.

2. If \( c, c' : X \times I(X, \varepsilon) \hookrightarrow W \) are collars, there is \( 0 < \varepsilon \leq \varepsilon' \) and a homotopy of collars \( C : X \times I(X, \varepsilon) \times [0, 1] \to W \) joining \( c|_{X \times I(X, \varepsilon)} \) to \( c'|_{X \times I(X, \varepsilon)} \).

3. If \( C : X \times I(X, \delta) \times [0, 1] \to W \) is a homotopy of collars, there is a collar \( \tau : X \times I(X, \delta) \hookrightarrow W \) with

\[
\begin{align*}
\tau|_{X \times I(X, \delta/3)} &= C_1|_{X \times I(X, \delta/3)}, \\
\tau|_{X \times (I(X, \delta) \cap I(X, 2\delta/3))} &= C_0|_{X \times (I(X, \delta) \cap I(X, 2\delta/3))}.
\end{align*}
\]

**Proof.**

1. Using a partition of unity, one constructs on an open \( U \subset W \) containing \( X \) a vector field \( w \in \mathfrak{X}(U) \) with \( w \) pointing inwards if \( X \subset \partial W \), and \( w \) in the positive coorientation if \( X \) is interior.

Let \( \phi : U \times \mathbb{R} \supset \text{dom}(\phi) \to U \) denote the local flow of \( w \), and choose any embedding \( \psi : X \times I(X, \varepsilon) \hookrightarrow \text{dom}(\phi) \) with \( \psi|_{X \times \{0\}} \) the inclusion \( X \hookrightarrow \text{dom}(\phi) \). Then \( c := \phi \circ \psi \) is a collar.

2. Define \( v_s := (1-s)v + sv' \in \mathfrak{X}(U) \), for \( s \in [0, 1] \), and let \( \phi_{v_s} : U \times \mathbb{R} \supset \text{dom}(\phi_{v_s}) \to U \) denote the local flow of \( v_s \). Choose a homotopy of embeddings \( \psi_s : X \times I(X, \varepsilon) \hookrightarrow \text{dom}(\phi_{v_s}) \), \( 0 \leq s \leq 1 \), with \( \psi_s|_X \) the inclusion, and set \( C_s := \phi_{v_s} \circ \psi_s : X \times I(X, \varepsilon) \to W \).

3. Let \( s \mapsto w_s \) denote the time-dependent vector field \( \frac{d}{dt} \in \mathfrak{X}(\text{im}(C_s)) \), and note that \( w_s(x) = 0 \) for all \( x \in X \) and \( s \in [0, 1] \); hence \( w_s \) has bounded velocity on some \( U' \supset X \). Choose then a smooth function \( \varrho : X \times I(X, \varepsilon) \times [0, 1] \to \mathbb{R} \), with \( \varrho_s = 1 \) on a smaller open \( U''_s \subset U'_s \) around \( X \), and set \( \varpi_s := \varrho_s w_s \in \mathfrak{X}(W) \). Then \( \varpi \) has bounded velocity, and thus generates an isotopy \( \phi^s \) of \( W \) with \( d_s \phi_s = \text{id} \) for all \( x \in X \) and \( s \in [0, 1] \), and \( \phi^1 C_0 \) agrees with \( C_0 \) away from \( X \), and with \( C_1 \) around it.
Corollary 4. Suppose $W,W'$ are smooth manifolds with boundary, that $X \subset \partial W$ be a sum of outgoing connected, and that $h : X \hookrightarrow \partial W'$ embeds $X$ as a sum of incoming connected components of $\partial W'$. Then the topological space $W \cup_h W'$ carries a canonical structure of smooth manifold with boundary, and

$$\partial(W \cup_h W') = (\partial W \setminus X) \bigsqcup (\partial W' \setminus h(X))$$

Proof. Suppose for simplicity that $X$ is connected; the general case is argued component-by-component.

We need first introduce a smooth structure on $W \cup_h W'$. Choose collars $c : X \times (-\varepsilon,0) \hookrightarrow W; \quad c' : h(X) \times [0,\varepsilon) \hookrightarrow W'$ and define the space $W \cup_h,c W'$ according to the diagram

$$X \times ((-\varepsilon,\varepsilon) \setminus 0) \xrightarrow{H} (W \setminus X) \bigsqcup (W' \setminus h(X))$$

where

$$H(x,t) = \begin{cases} c(x,t) & \text{if } t < 0; \\ c'(h(x),t) & \text{if } t > 0. \end{cases}$$

This exhibits $W \cup_h,c W'$ as a smooth manifold with the boundary as in the statement.

We need now show that the recipe above is independent of the choices of collars $c,c'$ up to a diffeomorphism.

So suppose $\gamma,\gamma'$ are two different choices of collars, and let $W \cup_{h,\gamma} W'$ denote the manifold arising from those choices. Then note that the identity maps $\text{id}_W,\text{id}_{W'}$, glue to a homeomorphism $G : W \cup_{h,\gamma} W' \to W \cup_{h,\gamma'} W'$.

On the $X \times (-\varepsilon,+\varepsilon)$ part of those manifolds, $G$ reads

$$G = \begin{cases} \gamma c^{-1} & \text{on im } c; \\ \gamma' c'^{-1} & \text{on im } c'. \end{cases}$$

According to Lemma 11, $c,c'$ can be modified to a collars $\overline{c},\overline{c}'$, with

$$\overline{c} = \begin{cases} c & \text{on } X \times (-\varepsilon/3,0]; \\ \gamma & \text{on } X \times (-\varepsilon,-2\varepsilon/3). \end{cases}, \quad \overline{c}' = \begin{cases} c' & \text{on } h(X) \times [0,\varepsilon/3); \\ \gamma' & \text{on } h(X) \times (2\varepsilon/3,\varepsilon). \end{cases}$$

We then modify $G$ to a diffeomorphism $\overline{G} : W \cup_{h,\gamma} W' \to W \cup_{h,\gamma'} W'$,

$$\overline{G} := \begin{cases} G & \text{outside } X \times (-\varepsilon,\varepsilon); \\ \gamma \overline{c}^{-1} & \text{on im } c; \\ \gamma' \overline{c}'^{-1} & \text{on im } c'. \end{cases}$$

Definition 14. We refer to $W \cup_h W'$ as the concatenation of $W,W'$ along $h$.

Example 6. Let $W$ be any manifold with boundary, and define $2W := W \cup_{\text{id}_W} W$, the double of $W$. Note that $\partial(2W) = \emptyset$. 

Note that, by its very construction, concatenation is 'distributive', in the sense that if we are given a further manifold with boundary \( W'' \), \( Y \subset \partial W' \) is outgoing, and \( h' : Y \rightarrow \partial W'' \) is an incoming embedding, then there is a natural identification
\[
(W \cup_h W') \cup_{h'} W'' \cong W \cup_h (W' \cup_{h'} W'').
\]

**Definition 15.** A factorization of a manifold with boundary \( W \) is a presentation as a concatenation of manifolds with boundary:
\[
W = W_0 \cup_{h_1} W_1 \cup_{h_2} \cdots \cup_{h_k} W_k.
\]

**Lemma 12.** Every cobordism \( \mathcal{C} \) can be factorized into elementary cobordisms.

**Proof.** Let \( \mathcal{C} = (W ; M_0, M_1) \) be a cobordism. Double \( W \) to the manifold (without boundary) \( 2W \), and note that \( \partial W \) embeds as a compact submanifold of \( 2W \).

Choose any \( f' : 2W \rightarrow [-1, +1] \) with \( f' \cap \partial \mathbb{D}^1 \) and \( f' + 1 \partial \mathbb{D}^1 = \partial W \). Use Lemma 10 to perturb \( f' \) to \( f'' \in \text{Morse}_\#(2W) \); choose \( f'' \) so \( \mathcal{C}^1 \)-close to \( f' \) so that \( \partial \mathbb{D}^1 \) are regular values for \( f_1' := (1 - t)f' + tf'', \) \( 0 \leq t \leq 1 \). Then there is a homotopy of embeddings \( \psi : \partial W \times [0, 1] \rightarrow 2W \) tracking \( f_1'^{-1} \partial \mathbb{D}^1 \):
\[
f_1' \psi_t(\partial_i W) = i, \quad i = 0, 1.
\]

By the Isotopy Extension Lemma 13 below, \( \psi \) can be extended to an isotopy \( \varphi : 2W \times [0, 1] \rightarrow 2W \); then
\[
f := f'' \circ \varphi_1|_{2W \setminus (W \setminus \mathbb{D})} \in C^\infty(W)
\]
is transverse to \( \partial \mathbb{D}^1 \) and pulls it back to \( \partial W \), and is a non-resonant Morse function in the interior of \( W \). Now choose \( a_i \in \mathbb{R} \setminus \text{Crit}(f) \) such that every \( c \in \text{Crit}(f) \) lies in exactly one interval \( (a_i, a_{i+1}) \); then the concatenation of the cobordisms
\[
C_i := (W_i, f^{-1}(a_{i-1}), f^{-1}(a_i))
\]
is diffeomorphic to \( W \).

**Lemma 13** (Isotopy Extension Lemma). Let \( W \) be a manifold with boundary, and \( X \subset W \) a closed submanifold, with either \( X \subset (W \setminus \partial W) \) or \( X \subset \partial W \). Then every homotopy of embeddings \( \psi : X \times [0, 1] \rightarrow W \), \( \psi_t : X \rightarrow W \), whose velocity \( \frac{d\psi_t}{dt} \) is bounded, extends to an isotopy \( \varphi : W \times [0, 1] \rightarrow W \).

**Proof.** **Case 1 :** \( X \subset (W \setminus \partial W) \).

Consider
\[
\hat{\psi} : X \times [0, 1] \rightarrow W \times [0, 1], \quad \hat{\psi}(x,t) = (\psi_t(x), t).
\]
The hypotheses ensure that \( \hat{\psi} \) is a closed embedding, and that
\[
\wbar{\psi} := \frac{d\psi_t}{dt} + \partial / \partial t
\]
is defined along its image and has bounded velocity.

Choose :
- a tubular neighborhood
  \[
  (W \setminus \partial W) \times I \supset E \xrightarrow{p} \hat{\psi}(X \times [0, 1]);
  \]
- a smooth function \( g \in C^\infty(E) \), with \( g = 1 \) around \( \hat{\psi}(X \times [0, 1]) \), and whose support meets every fibre of \( p \) in a compact set;
- an Ehresmann connection \( \text{hor} : \mathcal{X}(\hat{\psi}(X \times [0, 1])) \rightarrow \mathcal{X}(E) \).

Then set \( w := \text{ghor}(\wbar{\psi}) \in \mathcal{X}(W \times [0, 1]) \) and observe that \( w = w_t + f \partial / \partial t \), where \( w_t \in \mathcal{X}(W) \) is supported in the interior of \( W \), and extends \( \frac{d\psi_t}{dt} \). Hence \( w_t \) gives rise to an isotopy of \( W \) extending \( \psi \).

**Case 2 :** \( X \subset \partial W \).

Apply Case 1 twice, first to \( X \subset \partial W \), and then to \( \partial W \subset W \).
3.3. Exercises.

(1) Show that $J_k(M, M')$ is indeed a smooth manifold, and compute its dimension.

(2) Show that $j_k : C^k(M, M') \to C^0(M, J_k(M, M'))$ is continuous in both the weak and the strong topologies, and has closed image in the weak topology.

(3) Let $M \subset \mathbb{R}^N$ be a submanifold. For each $y \in \mathbb{R}^N$, let $f_y : M \to \mathbb{R}$ denote $x \mapsto ||y - x||^2$. Show that for $y$ generic, $f_y \in \text{Morse}(M)$ (meaning that the set of points for which the stated property holds is residual).

(4) Compute $\pi_n(S^m)$ for all $m > n \geq 0$.

(5) Two compact manifolds $M_0^m, M_1^m$ are called (oriented) cobordant if there exists a (oriented) cobordism $C = (W; M_0, M_1)$. Show that:
   (a) Being (oriented) cobordant to is an equivalence relation.
   (b) The sets $N_m$, $\Omega_m$ of equivalence classes under cobordism and oriented cobordism relations, respectively, are abelian groups under disjoint union $\bigsqcup$.
   (c) If $f, f' : M \to M'$ are homotopic, and transverse to a closed submanifold $X \subset M'$, then $f^{-1}X$ and $f'^{-1}X$ are cobordant. If $M, M'$ and $X$ are orientable, $f^{-1}X$ and $f'^{-1}X$ are oriented cobordant.
   (d) Compute $N_i$ and $\Omega_i$, for $i = 0, 1$.

(6) Let $M, M'$ be compact smooth manifolds, and let $G := \text{Diff}(M') \times \text{Diff}(M)$ act on $C^\infty(M, M')$ by

$$(\psi, \varphi) : f \mapsto \psi \circ f \circ \varphi^{-1},$$

where $G$ is endowed with the $C^\infty$ topology. A map $f$ is called stable if every $f'$ close enough to $f$ lies in the same orbit as $f$.

Show that $f \in C^\infty(M, \mathbb{R})$ is stable only if $f \in \text{Morse}_\varphi(M)$.\footnote{We will see later that $\text{Morse}_\varphi(M)$ is precisely the space of stable functions on $M$.}
4.3. Surgery. For every $1 \leq \lambda < m$, consider the “standard” diffeomorphisms

\[
\text{std}_\lambda : S^{\lambda-1} \times (\mathbb{D}^{m-\lambda+1} \setminus 0) \simeq (\mathbb{D}^{m-\lambda} \setminus 0) \times S^{m-\lambda}
\]

\[
\text{std}_\lambda : (u, \theta v) \mapsto (\theta u, v), \quad (u, v) \in S^{\lambda-1} \times S^{m-\lambda}, \quad \theta \in (0, 1).
\]

Fix an embedding

\[
\varphi : S^{\lambda-1} \times \mathbb{D}^{m-\lambda+1} \hookrightarrow M^m,
\]

and consider the smooth manifold \( \text{Surg}(M, \varphi) \) defined by the pushout diagram

\[
\begin{array}{ccc}
S^{\lambda-1} \times (\mathbb{D}^{m-\lambda+1} \setminus 0) & \xrightarrow{\varphi} & M \setminus \varphi(S^{\lambda-1}) \\
\text{std}_\lambda \downarrow & & \downarrow \\
\mathbb{D}^{\lambda} \times S^{m-\lambda} & \rightarrow & \text{Surg}(M, \varphi).
\end{array}
\]

Observe that \( \text{Surg}(M, \varphi) \) comes equipped with a canonical embedding \( \text{Surg}(\varphi) : \mathbb{D}^{\lambda} \times S^{m-\lambda} \hookrightarrow \text{Surg}(M, \varphi) \). Producing \( \text{Surg}(M, \varphi) \) out of \( M \) has the effect of removing a \((\lambda-1)\)-sphere, embedded with trivial normal bundle in \( M \), and replacing it by a \((m-\lambda)\)-sphere, also embedded with trivial normal bundle.

**Definition 16.** We say that \( \text{Surg}(M, \varphi) \) is obtained from \( M \) by a surgery of type \( \lambda \).

**Lemma 14.** If \( \varphi_t : S^{\lambda-1} \times \mathbb{D}^{m-\lambda+1} \hookrightarrow M \) is a homotopy of embeddings, then \( \text{Surg}(M, \varphi_0) \simeq \text{Surg}(M, \varphi_1) \).

**Proof.** Extend \( \frac{d \varphi_t}{dt} \in X(\text{im} \varphi_t) \) to a globally defined (time-dependent) vector field \( w_t \in X(M) \). We can further demand that the support of \( w_t \) be a small neighborhood of \( \text{im} \varphi_t \). Denote by \( \phi^t \) the isotopy it generates, and observe that

\[
\phi^t(\varphi_t(u, \theta v)) = \varphi_t(u, \theta v).
\]

Then

\[
\phi^1 \prod \text{id} : (M \setminus \varphi_0(S^{\lambda-1})) \prod \mathbb{D}^{\lambda} \times S^{m-\lambda} \simeq (M \setminus \varphi_1(S^{\lambda-1})) \prod \mathbb{D}^{\lambda} \times S^{m-\lambda}
\]

descends to a diffeomorphism \( \text{Surg}(M, \varphi_0) \simeq \text{Surg}(M, \varphi_1) \). \( \square \)

4.2. A closer look at model singularities. Let \( L_{\lambda} \subset \mathbb{R}^{\lambda} \times \mathbb{R}^{m-\lambda+1} \) be the subspace defined by

\[
L_{\lambda} := \{(x, y) : -1 \leq Q_{\lambda}(x, y) \leq +1, |x| |y| < \sinh 1 \cosh 1 \},
\]

where as usual \( Q_{\lambda} \) denotes \( Q_{\lambda}(x, y) = -|x|^2 + |y|^2 \).

Note that \( L_{\lambda} \) is a smooth manifold with boundary \( \partial L_{\lambda} = \partial_{\text{left}} L_{\lambda} \prod \partial_{\text{right}} L_{\lambda} \), where

\[
\partial_{\text{left}} L_{\lambda} := \{(x, y) \in L_{\lambda} : Q_{\lambda}(x, y) = -1\}
\]

\[
\partial_{\text{right}} L_{\lambda} := \{(x, y) \in L_{\lambda} : Q_{\lambda}(x, y) = +1\}.
\]

We let \( \mathbb{R}_{\infty} \) denote

\[
\mathbb{R}_{\infty} := (\mathbb{R}^{\lambda} \setminus 0) \times (\mathbb{R}^{m-\lambda+1} \setminus 0) \subset \mathbb{R}^{\lambda} \times \mathbb{R}^{m-\lambda+1}.
\]
Lemma 15. There exist diffeomorphisms

\[\varphi_{\text{left}} : S^{\lambda - 1} \times D^{m - \lambda + 1} \cong \partial_{\text{left}} L_\lambda\]

\[\varphi_{\text{right}} : D^{\lambda} \times S^{m - \lambda} \cong \partial_{\text{right}} L_\lambda\]

\[\text{std}^\lambda : \partial_{\text{left}} L_\lambda \cap \mathbb{R} \cong \partial_{\text{right}} L_\lambda \cap \mathbb{R},\]

such that

\[\partial_{\text{left}} L_\lambda \cap \mathbb{R} \xrightarrow{\varphi_{\text{left}}^{-1}} M \setminus \varphi(S^{\lambda - 1})\]

\[\partial_{\text{right}} L_\lambda \xrightarrow{\varphi_{\text{right}}^{-1}} \text{Surg}(M, \varphi)\]

\[\partial_{\text{right}} L_\lambda \cap \mathbb{R} \xrightarrow{\text{Surg}(\varphi)\varphi_{\text{right}}^{-1}} \text{Surg}(M, \varphi) \setminus \text{Surg}(\varphi)(S^{m - \lambda})\]

\[\partial_{\text{left}} L_\lambda \cap \mathbb{R} \xrightarrow{\varphi_{\text{left}}^{-1}} \partial_{\text{right}} L_\lambda \cap \mathbb{R} \xrightarrow{\varphi_{\text{right}}^{-1}} M\]

Proof. Define \(\text{std}^\lambda : \mathbb{R} \cong \mathbb{R}\) by the formula

\[\text{std}^\lambda : (x, y) \mapsto \left(\frac{|x|}{|y|}, x, \frac{|y|}{|x|}y\right),\]

and observe that \(\text{std}^\lambda\) is an involution, \(\text{std}^\lambda = (\text{std}^\lambda)^{-1}\). Moreover, it induces a diffeomorphism

\[\partial_{\text{left}} L_\lambda \cap \mathbb{R} \cong \partial_{\text{right}} L_\lambda \cap \mathbb{R},\]

which we still denote by \(\text{std}^\lambda\).

Now define the diffeomorphisms

\[\varphi_{\text{left}} : S^{\lambda - 1} \times D^{m - \lambda + 1} \cong \partial_{\text{left}} L_\lambda, \quad \varphi_{\text{left}}(u, \theta v) = (u \cosh \theta, v \sinh \theta)\]

\[\varphi_{\text{right}} : D^{\lambda} \times S^{m - \lambda} \cong \partial_{\text{right}} L_\lambda, \quad \varphi_{\text{right}}(\theta u, v) = (u \sinh \theta, v \cosh \theta)\]

Then

\[S^{\lambda - 1} \times (\overline{D}^{m - \lambda + 1} \setminus 0) \xrightarrow{\varphi_{\text{left}}} \partial_{\text{left}} L_\lambda \cap \mathbb{R}\]

\[\overline{D}^{\lambda} \setminus 0 \times S^{m - \lambda} \xrightarrow{\varphi_{\text{right}}} \partial_{\text{right}} L_\lambda \cap \mathbb{R}\]

commutes. Hence

\[\partial_{\text{left}} L_\lambda \cap \mathbb{R} \xrightarrow{\varphi_{\text{left}}^{-1}} M \setminus \varphi(S^{\lambda - 1})\]

\[\partial_{\text{right}} L_\lambda \xrightarrow{\varphi_{\text{right}}^{-1}} \text{Surg}(M, \varphi)\].
is also a pushout diagram. On the other hand, the pushout of the outer diagram in

\[
\begin{array}{c}
\mathbb{S}^{\lambda-1} \times (\mathbb{B}^{m-\lambda+1} \setminus 0) \\
\downarrow \text{std} \\
(\mathbb{B}^{\lambda} \setminus 0) \times \mathbb{S}^{m-\lambda} \\
\downarrow \text{std} \\
\mathbb{S}^{\lambda-1} \times (\mathbb{B}^{m-\lambda+1} \setminus 0)
\end{array}
\xrightarrow{\varphi_{\text{left}}} \partial_{\text{left}} L_\lambda \cap \mathbb{R}_x 
\xrightarrow{\psi_{\text{left}}} M \setminus \varphi(\mathbb{S}^{\lambda-1})
\xrightarrow{\varphi_{\text{right}}^{-1}} \partial_{\text{right}} L_\lambda \cap \mathbb{R}_x 
\xrightarrow{\psi_{\text{right}}} \text{Surg}(\mathbb{S}^{\lambda-1} \setminus 0) 
\xrightarrow{\varphi_{\text{right}}} \text{Surg}(M, \varphi) \setminus \text{Surg}(\varphi)(\mathbb{S}^{m-\lambda})
\]

is clearly \(M \setminus \varphi(\mathbb{S}^{\lambda-1})\), as the top horizontal arrow equals \(\varphi\) and the left vertical one is identical. Hence

\[
\begin{array}{c}
\partial_{\text{right}} L_\lambda \cap \mathbb{R}_x \\
\downarrow \text{std} \\
\partial_{\text{left}} L_\lambda \\
\downarrow \text{std} \\
M
\end{array}
\xrightarrow{\text{Surg}(\varphi)\psi_{\text{right}}^{-1}} \text{Surg}(M, \varphi) \setminus \text{Surg}(\varphi)(\mathbb{S}^{m-\lambda})
\]

\[\square\]

**Theorem 11.** There is an elementary cobordism \((\mathcal{C}, f)\) of index \(\lambda\) between \(M\) and \(\text{Surg}(M, \varphi)\).

**Proof.** For every \((x, y) \in L_\lambda\), the curve

\[t \mapsto (tx, t^{-1}y), t > 0,\]

is orthogonal to the level sets \(Q_\lambda = c, c \neq 0\).

Observe that

\[t = t(x, y) := \sqrt{1 + \frac{1}{2|x|^2}x^2y^2} \implies Q_\lambda(tx, t^{-1}y) = -1;\]

hence we obtain a diffeomorphism

\[
\begin{align*}
\psi : L_\lambda \cap \mathbb{R}_x &\cong (\partial_{\text{left}} L_\lambda \cap \mathbb{R}_x) \times [-1, +1], \\
\psi : (x, y) &\mapsto ((t(x, y)x, t(x, y)^{-1}y), Q_\lambda(x, y)).
\end{align*}
\]

We can thus form the smooth manifold \(W\) by

\[
\begin{array}{c}
L_\lambda \cap \mathbb{R}_x \\
\downarrow \text{std} \\
L_\lambda \\
\downarrow \text{std} \\
W
\end{array}
\xrightarrow{(\psi_{\text{left}}^{-1} \times \text{id})\psi} (M \setminus \varphi(\mathbb{S}^{\lambda-1}) \times [-1, +1]
\]

and note that

\[
\partial W = \partial_0 W \coprod \partial_1 W,
\]

where

\[
\begin{array}{c}
\partial_{\text{left}} L_\lambda \cap \mathbb{R}_x \\
\downarrow \text{std} \\
\partial_{\text{left}} L_\lambda \\
\downarrow \text{std} \\
\partial_0 W
\end{array}
\xrightarrow{\varphi_{\text{left}}} (M \setminus \varphi(\mathbb{S}^{\lambda-1})
\]

\[
\xrightarrow{\varphi_{\text{right}}} \partial_1 W
\]

and
and
\[ \partial_{\text{right}} L_\lambda \cap \mathbb{R}_x \to (M \setminus \varphi(S^{\lambda-1})) \]
\[ \partial_{\text{right}} L_\lambda \to \partial_0 W \]

so \( \partial_0 W \simeq M \) and \( \partial_1 W \simeq \text{Surg}(M, \varphi) \).

Hence \( W \) is a cobordism between \( M \) and \( \text{Surg}(M, \varphi) \); to finish we must indicate the pertinent elementary Morse function \( f \in \text{Morse}(W) \). But observe that under the above identifications, the smooth map
\[
\tilde{f} : (M \setminus \varphi(S^{\lambda-1}) \times [-1, +1] \coprod_{\varphi(S^{\lambda-1})} \coprod L_\lambda \to \mathbb{R}
\]
\[
\tilde{f}|(M \setminus \varphi(S^{\lambda-1}) \times [-1, +1]) = \text{pr}_2, \quad \tilde{f}|L_\lambda = Q_\lambda
\]
descends to a smooth \( f \in C^\infty(W) \) with the required properties.

On the other hand, suppose \((\mathcal{C}, f)\) is an elementary cobordism, where \( f : W \to \mathbb{D}^3 \) is an elementary Morse function with a unique critical point \( p \) of index \( \lambda \) at the level set 0.

We wish to define an embedding
\[
\varphi : S^{\lambda-1} \times \mathbb{D}^m \to \partial_0 W
\]
Fix a Morse chart \( e : B_{2\varepsilon}^{m+1} \hookrightarrow W^{m+1} \) centred at \( p \),
\[
e(0) = p \in \text{Crit}(f), \quad e^* f = Q_\lambda.
\]

Then
\[
\varphi' : S^{\lambda-1} \times \mathbb{D}^m \hookrightarrow f^{-1}(\varepsilon),
\]
\[
(u, \theta) \mapsto e(\varepsilon u \cosh \theta, \varepsilon u \sinh \theta)
\]
embeds \( S^{\lambda-1} \times \mathbb{D}^m \) in the regular level set \( f = \varepsilon \). The (local) flow \( \phi^t \) of the vector field \( w := \frac{\nabla f}{\|\nabla f\|} \in X(M \setminus p) \), \( \phi : (M \setminus p) \times \mathbb{R} \supset \text{dom}(\phi) \to M \setminus p \), determines a homotopy of embeddings
\[
\varphi'_1 : S^{\lambda-1} \times \mathbb{D}^m \to W,
\]
\[
S^{\lambda-1} \times \mathbb{D}^m \times [0, 1 - \varepsilon] \to W, \quad ((u, \theta) \cdot (t)) \mapsto \phi^{-t}(\varphi'(u, \theta)),
\]
and we set
\[
\varphi := \varphi'_1 - \varepsilon : S^{\lambda-1} \times \mathbb{D}^m \to \partial_0 W
\]
Observe that the choice of \( \varepsilon > 0 \) is immaterial, since the embeddings determined by any two choices according to the recipe above must coincide.

By the same token, we can drag the embedding
\[
\Phi' : S^{\lambda-1} \times \mathbb{D}^m \to f^{-1}(\varepsilon), \quad (u, \theta) \mapsto e(\varepsilon u \sinh \theta, \varepsilon u \cosh \theta)
\]
along the flow of \( w \) from time \( t = 0 \) to \( t = 1 - \varepsilon \) to obtain an embedding
\[
\Phi : S^{\lambda-1} \times \mathbb{D}^m \to \partial_1 W.
\]

**Definition 17.** We call the embeddings \( \varphi, \Phi \) characteristic- and cocharacteristic embeddings of \((\mathcal{C}, f)\).

**Remark 3.** Note that the (co)-characteristic embedding depends on the choice of Morse chart, and also on the vector field \( \nabla f \) which we used to drag objects around. Such choices will be implicit whenever we speak of such embeddings.
Theorem 12. If \((C, f)\) is elementary of index \(\lambda\), then \(\partial_1 W \simeq \text{Surg}(\partial_0 W, \varphi)\), for some characteristic embedding \(\varphi : S^{\lambda-1} \times D^{m-\lambda+1} \to \partial_0 W\).

Proof. In terms of the notation above, one argues as in Theorem 11 to deduce that \(f^{-1}(\varepsilon) \simeq \text{Surg}(f^{-1}(-\varepsilon), \varphi')\), and \(\partial_0 W \simeq f^{-1}(-\varepsilon), \partial_1 W \simeq f^{-1}(\varepsilon)\) under \(\phi^k(v, \sigma-1)\).

Let \((C, f)\) be an elementary cobordism of index \(\lambda\), with characteristic and cocharacteristic embeddings \(\varphi, \Phi\), respectively.

Definition 18. The core disk \(\text{Core}_\lambda(p)\) of the critical point \(p\) is the union of trajectories of \(\nabla f\) beginning in \(\varphi(S^{\lambda-1}) \subset \partial_0 W\) and ending in \(p\).

Its cocore disk \(\text{Cocore}^{m-\lambda}(p)\) is the union of trajectories of \(\nabla f\) beginning in \(p\) and ending in \(\Phi(S^{m-\lambda}) \subset \partial_1 W\).

Note that it follows from the above discussion that these are smoothly embedded disks, meeting transversally at \(p\), and determining the decomposition

\[ T_p W = T_p \text{Core}_\lambda(p) \oplus T_p \text{Cocore}^{m-\lambda}(p) \]

into negative-definite and positive-definite subspaces for \(\text{Hess}_p(f)\).

Corollary 5. If \((C, f)\) be an elementary cobordism of index \(\lambda\),

\[(\partial_0 W \cup \text{Core}_\lambda(p)) \to \partial_1 W\]

is a deformation retraction. In particular

\[ H_\bullet(W, \partial_1 W; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = \lambda; \\ 0 & \text{otherwise.} \end{cases} \]

and so the index of an elementary cobordism \(C\) is independent of the choice of elementary Morse function.

4.2.1. Exercises.

1. A gradient-like vector field for \(f \in \text{Morse}(M)\) is a \(w \in \mathfrak{X}(M)\) such that:
   - \(w > 0\) on \(M \setminus \text{Crit}(f)\);
   - For every \(p \in \text{Crit}(f)\), there is a Morse chart \(e : B_{2x} \hookrightarrow M\) centred at \(p\), pulling \(w\) back to
     \[ e^* w = -2 \sum_{i=1}^\lambda x_i \frac{\partial}{\partial x_i} + 2 \sum_{i=1}^n y_i \frac{\partial}{\partial y_i}. \]
     (a) Convince yourself that, except for Lemma 4, all arguments involving the gradient \(\nabla f\) with respect to some Riemannian metric remain true if \(\nabla f\) is replaced by a gradient-like vector field \(w\).
   (b) Let \(w\) be a gradient-like vector field for \(f\) Morse on the compact manifold \(M\), and let \(\varphi : M \times \mathbb{R} \to M\) denote its flow. For any \(x \in M\), let \(\omega(x)\) be the collection of those points of \(M\) which are limit points sequences of the form \((\phi^n(x))_{n \geq 0}\), where \(t_n \to +\infty\). Show that \(\omega(x)\) is contained in a level set of \(f\). Similarly, the limit points \(\alpha(x)\) to sequences of the form \((\phi^k(x))_{n \geq 0}\), \(t_n \to -\infty\), lie in a single level set of \(f\).
   (c) Show that \(\alpha(x)\) and \(\omega(x)\) are invariant under the flow of \(w\).
   (d) Show that \(\alpha(x) \subset \text{Crit}(f) \supset \omega(x)\).
   (e) Show that \(\alpha(x) = \{p\}\) and \(\omega(x) = \{q\}\). Conclude that, for every \(x \in M\), \(\lim_{t \to \pm \infty} \phi^t(x)\) exists and is a critical point.

2. Prove Corollary 5.
REFERENCES


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