Utrecht, L1

M = manifold, G = (connected) Lie group
$\mathfrak{g}$ = Lie algebra of G

Def $(M, \omega \in \Omega^2(M), \mu : M \to \mathfrak{g}^*)$ is a Hamiltonian G-space if

① G acts on M.

② ω is symplectic, that is

a $d\omega = 0$ b $\omega_m : T_mM \times T_mM \to \mathbb{R}$ nondegenerate $\forall m$

$\Updownarrow$

$\omega_m^\flat : T_mM \to T_m^*M$ iso

③ $\forall x \in \mathfrak{g}$, $\boxed{\iota(x_M)\,\omega = d\langle \mu, x\rangle}$

action: $\mathfrak{g} \longrightarrow \mathfrak{X}(M)$, $x \longmapsto x_M$

Remark: ③ $\Rightarrow$ ω is G-invariant:

$L(x_M)\,\omega = (d\circ\iota(x_M) + \iota(x_M)\circ d)\,\omega = d\,d\langle\mu, x\rangle = 0$

④ $\mu : M \longrightarrow \mathfrak{g}^*$ is G-equivariant

Remark for G semisimple,
ω G-invariant $\Rightarrow$ $\exists$ moment map μ,
μ is unique, and it is equivariant

Ex: · coadjoint orbits
$\xi \in \mathfrak{g}^*$; $\mathcal{O}_\xi = \{ Ad_g^* \xi \,;\, g \in G \}$

Thm $\exists!\ \omega \in \Omega^2(\mathcal{O}_\xi)$ s.t. $(\mathcal{O}_\xi, \omega, \mu : \mathcal{O}_\xi \hookrightarrow \mathfrak{g}^*)$
is a Hamiltonian G-space

$\eta \in \mathcal{O}_\xi$; $\omega_\eta(x_M, y_M) = \langle \eta, [x,y]\rangle$ Kirillov 2-form

Notation G - Lie group
$\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ - left-invariant and right invariant MC forms
$\theta^L = g^{-1}dg \quad \theta^R = dg\, g^{-1}$

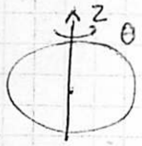
structure equations $d\theta^L = \cancel{g^{-1}}\, d(g^{-1}dg) = -g^{-1}dg\, g^{-1}dg = -\frac{1}{2}[\theta^L, \theta^L]$
$d\theta^R = \frac{1}{2}[\theta^R, \theta^R]$

Pbm $\Pi_\xi : G \longrightarrow \mathcal{O}_\xi \,,\quad g \mapsto Ad_g^* \xi$

$$\Pi_\xi^* \omega = -d \langle \xi, \theta^L \rangle = -d \langle \eta, \theta^R \rangle$$

Ex $G = SU(2) \Rightarrow \mathfrak{g}^* \cong \mathbb{R}^3 \circlearrowleft SO(3)$

orbits = origin + spheres of radius r



$$\omega = d\theta \wedge dz \,,\quad \iota\left(\frac{\partial}{\partial \theta}\right)\omega = dz$$

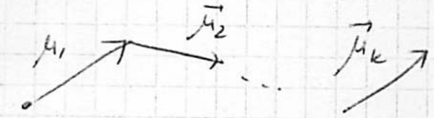
$$Vol = \int d\theta \wedge dz = 2\pi \cdot 2r = 4\pi r$$

Ex : products (M_i, ω_i, μ_i), $i = 1, 2$

$M = M_1 \times M_2$, $\omega = \pi_1^* \omega_1 + \pi_2^* \omega_2$, $\mu(a, b) = \mu_1(a) + \mu_2(b)$

$r_1, \dots, r_k \in \mathbb{R}_{>0}$ $\quad M = \underbrace{S^2 \times \dots \times S^2}_{k \text{ times}}$ $\quad \omega = \pi_1^* \omega_{r_1} + \dots + \pi_k^* \omega_{r_k}$

$\mu = \mu_1 + \dots + \mu_k =$



$\in \mathbb{R}^3$

Recall: <u>reduction</u>

M = Hamiltonian G-space $\Rightarrow M/G$ is a Poisson space

$C^\infty(M)^G$ carries a Poisson bracket

Thm (Marsden – Weinstein reduction)

(M, ω, μ) Hamiltonian G-space, $\xi \in \mathfrak{g}^*$

Assume that $G_\xi = \{ g \in G \,;\, Ad_g^* \xi = \xi \}$ acts *freely* on $\mu^{-1}(\xi)$

Then, $M_\xi = \mu^{-1}(\xi)/G_\xi$ is symplectic ~~w~~

$$\begin{array}{ccc} \mu^{-1}(\xi) & \xrightarrow{\ \iota\ } & M \\ \downarrow \pi & & \\ M_\xi & & \end{array} \,,\quad \underline{\pi^* \omega_\xi} = \iota^* \omega$$

injective $\Rightarrow \omega_\xi$ <u>unique</u>

Remark action quasi-free (discrete stabilizers) $\Rightarrow$ M_ξ is a symplectic orbifold $\Leftarrow$ ξ regular value

~~action~~ in general $\rightarrow$ M_ξ is a stratified symplectic space

Remark $\dim M_\xi = \dim M - \dim G - \dim G_\xi$

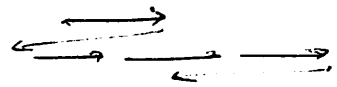
$\xi = 0 \quad$ ~~dim~~ $\Rightarrow G_\xi = G \Rightarrow \dim M_\xi = \dim M - 2 \dim G$

<u>Ex</u>: polygon spaces

$$M = \underbrace{S^2 \times \dots \times S^2}_{k \text{ times}} , \quad \omega = \pi_1^* \omega_{r_1} + \dots + \pi_k^* \omega_{r_k}$$

$$M_0 = \{ \mu_i \in \mathbb{R}^3 \; ; \; \|\mu_i\| = r_i , \; \mu_1 + \dots + \mu_k = 0 \} / SO(3)$$

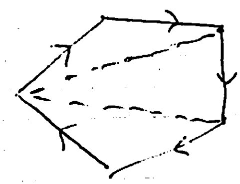
Stab trivial unless $\exists \, I \subset \{1, .. k\}$ s.t. $\sum_{i \in I} r_i = \sum_{i \notin I} r_i$



$$\dim M_0 = \dim M - 2 \dim G = 2k - 6 = 2(k-3)$$

<u>Pbm</u>: show that $\underbrace{\|\mu_1 + \mu_2\| , \|\mu_1 + \mu_2 + \mu_3\| , \dots \|\mu_1 + \dots + \mu_{k-2}\|}_{k-3}$ Poisson commute

what are the flows ?



<u>in general</u>: $\xi_1 , \dots , \xi_k \in \mathfrak{g}^*$

$M = \mathcal{O}_{\xi_1}$

<u>Example</u>: $\mathfrak{g} = U(n) \cong$ anti-Hermitian matrices

$\mathfrak{g}^* \cong \mathcal{H}_n$ Hermitian matrices $\langle \xi , x \rangle = \mathrm{Im} \, \mathrm{Tr}(x\xi)$

$\lambda = (\lambda_1 \geq \dots \geq \lambda_n) \in \mathbb{R}$ $\quad \mathcal{O}_\lambda = \{ \xi \in \mathcal{H}_n ; \text{ eigen } (\xi) = (\lambda_1 \geq \dots \geq \lambda_n) \}$

$M = \mathcal{O}_\lambda \times \mathcal{O}_\mu \times \mathcal{O}_\nu$ $\quad M_0 = \{ \xi, \eta, \zeta \in \mathcal{H}_n , \; \xi + \eta + \zeta = 0 , \; \xi \in \mathcal{O}_\lambda , \eta \in \mathcal{O}_\mu , \zeta \in \mathcal{O}_\nu \} / U(n)$

~~$M_0 \neq \emptyset$~~ ? Conditions on (λ, μ, ν) s.t. $M_0 \neq \emptyset$ Horn pbm

<u>Pbm</u> Solve Horn pbm for <u>$n = 2$</u>