Functions on $\mathcal{M}(\Sigma, G)$

$p \in \Sigma$, γ = closed curve based in p

$\mathrm{Hol}(A,\gamma) \in G$ can be viewed as a map $\mathcal{A}(\Sigma, G) \to G$

choose $\rho : G \to \mathrm{End}(V)$ a finite-dimensional rep of G

$$W^{\rho}_{\gamma}(A) := \mathrm{Tr}_V\, \rho\left(\mathrm{Hol}(A,\gamma)\right)$$

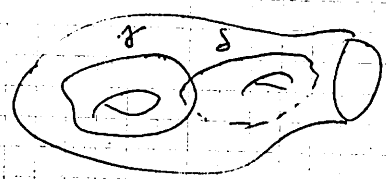
<u>Prop</u> $W^{\rho}_{\gamma}(A)$ is invariant under action of $\mathcal{G}(G)$

<u>Proof</u>

$$W^{\rho}_{\gamma}(A^{g}) = \mathrm{Tr}_V\, \rho\left(g(p)^{-1}\,\mathrm{Hol}(A,\gamma)\, g(p)\right) =$$

$$= \mathrm{Tr}_V\, \rho(g(p))^{-1}\, \rho(\mathrm{Hol}(A,\gamma))\, \rho(g(p)) = W^{\rho}_{\gamma}(A)$$

<u>Question</u>:



$$\{W^{\rho}_{\gamma}, W^{\rho'}_{\delta}\} = ?$$

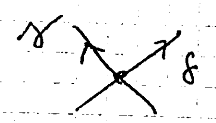
<u>Ex</u>: $G = GL(N)$ $\quad \rho : GL(N) \to \mathrm{End}(\mathbb{C}^N)$ the standard rep.

<u>Thm</u> (Goldman):

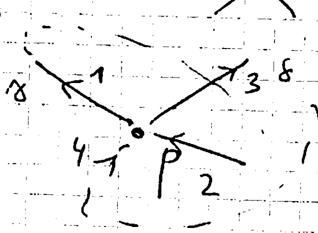
$$\{W_{\gamma}, W_{\delta}\} = \sum_{p \in \gamma \cap \delta} \varepsilon(\gamma,\delta;p)\; W_{\gamma *_p \delta}$$

assume intersections transverse

$\varepsilon(\gamma,\delta;p) = \pm 1 \Leftarrow$ orientation at the intersection



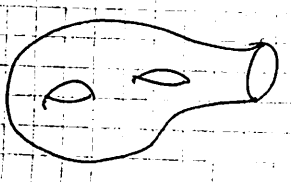
$\gamma *_p \delta$



product of γ and δ based at p

<u>Pbm</u> Use the Goldman bracket to show that

$W_{\gamma}(A)$ is central for $\gamma = \partial\Sigma \cong S^1$.



Combinatorial description of $\mathcal{M}(\Sigma, G)$

Prop Σ = simply connected, $A \in \Omega^1(\Sigma, \mathfrak{g})$ flat, $p \in \Sigma$
$\Rightarrow \exists!\ g : \Sigma \to G$ s.t. $g(p) = e$ and $\underset{\underset{g^{-1}dg}{\parallel}}{g^*\theta^L} = A$

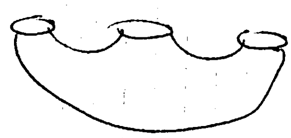
Thm $\mathcal{M}(\Sigma, G) \cong \mathrm{Hom}(\pi_1(\Sigma), G)/G$
↳ action by conjugation

$\mathcal{M}_p(\Sigma, G) \cong \mathrm{Hom}(\pi_1(\Sigma, p), G)$

Remark Σ = closed surface of genus g
$\pi_1(\Sigma)$ generated by $a_1, b_1, \ldots, a_g, b_g$ relation $[a_1, b_1] \ldots [a_g, b_g] = 1$
$\dim \mathcal{M}(\Sigma, G) = 2g \cdot \dim(G) - 2 \cdot \dim(G) = 2(g-1)\dim(G)$
for $g \geq 2$

Pbm (Jeffrey–Weitsmann) for $G = SU(2)$ find an completely integrable system on $\mathcal{M}(\Sigma, G)$ using the Goldman bracket

Ex: Σ =



$= S^2 \setminus 3$ holes

$\pi_1(\Sigma) = \langle a, b, c \rangle / abc = 1$

$\mathcal{M}(\Sigma, G) \longrightarrow \Delta_W^3$ Symplectic leaves:
Poisson space

$$\mathcal{M}(\Sigma, G, \alpha, \beta, \gamma) = \frac{\{(f, g, h) \in G^3;\ \pi(f) = \alpha,\ \pi(g) = \beta,\ \pi(h) = \gamma,\ fgh = 1\}}{G} \quad = \text{conjugation}$$

Pbm When $\mathcal{M}(\Sigma, G, \alpha, \beta, \gamma)$ is nonempty?
Similar to Horn problem

Solve it for: $G = SU(2)$

Ex: Σ =

$\pi_1(\Sigma) = \langle a, b, c \rangle / [a, b]c = 1$

symplectic leaves $\mathcal{M}(\Sigma, G, \alpha) = \dfrac{\{(f, g, h) \in G^3,\ \pi(h) = \alpha,\ [f, g]h = 1\}}{G}$

$\dim = 2\dim G + (\dim G - \dim G_\alpha) - 2\dim G =$
$= \dim G - \dim G_\alpha$

if the action is free

Computing symplectic forms

Consider Σ simply connected with boundary

$A \in \Omega^1(\Sigma, \mathfrak{g})$ flat $\Rightarrow \exists\, g : \Sigma \to G$ s.t. $A = g^{-1} dg$

Note: $\delta A = \delta(g^{-1} dg) = -g^{-1} d(\delta g \cdot g^{-1})\, g$

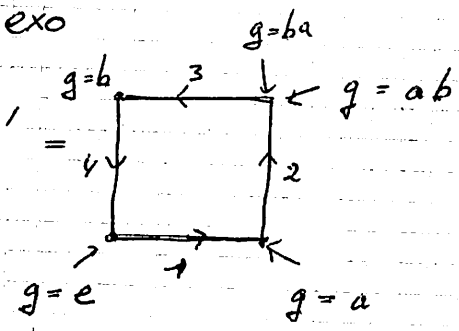
$$\Rightarrow \omega = \frac{1}{2}\int_\Sigma (\delta A \,\hat{,}\, \delta A) = \int_\Sigma \left(d(\delta g g^{-1}) \,\hat{,}\, d(\delta g g^{-1})\right) = \int_{\partial\Sigma} \left(\delta g g^{-1}, d(\delta g \cdot g^{-1})\right)$$

Prop Let $g : [a,b] \to G$, $g = fh$, $f = \text{const}$

$$\Rightarrow \int_{[a,b]} \left(\delta g g^{-1}, d(\delta g g^{-1})\right) = \left(f^{-1}\delta f, \delta h h^{-1}\right)\Big|_a^b + \int_{[a,b]} \left(\delta h h^{-1}, d(\delta h \cdot h^{-1})\right)$$

Proof $\leftarrow$ ~~exercise~~ exo

Ex: $\Sigma = \mathbb{T}^2$, $\Sigma' =$



A flat

$\Rightarrow$ $ab = ba$

$$\omega_1 + \omega_3 = -\left(b^{-1}\delta b, \delta a\, a^{-1}\right)$$

$$\omega_2 + \omega_4 = \left(a^{-1}\delta a, \delta b \cdot b^{-1}\right)$$

$$\Rightarrow \omega = \frac{1}{2}\left(\left(a^{-1}\delta a, \delta b \cdot b^{-1}\right) - \left(b^{-1}\delta b, \delta a \cdot a^{-1}\right)\right)$$

recall the coefficient

for (a,b) generic, $a = \exp(x)$, $b = \exp(y)$, $[x,y] = 0$

~~def~~ $\omega_{(a,b)} = (dx \,\hat{,}\, dy)$