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Lecture 3

We are now ready to move to the full theory of global aspects of Poisson manifolds. This global theory takes advantage of group like objects that can be associated to a Poisson manifold.

2. Group Like aspects

We have seen two Lie brackets associated w/ (M,π):

$$\left.\begin{array}{l} \cdot\ \{\cdot,\cdot\}: C^\infty(M)\times C^\infty(M)\to C^\infty(M) \\ \cdot\ [\cdot,\cdot]: \Omega^1(M)\times\Omega^1(M)\to\Omega^1(M)\end{array}\right\}\quad [df,dg]=d\{f,g\}$$

Q. Are there Lie group type objects integrating these Lie brackets?

A. Yes! They play a crucial role in various global problems associated with a Poisson manifold.

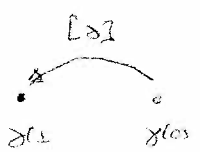
• Ordinary Geometry: what is the global object integrating the usual Lie bracket of vector fields? The usual answer is $\mathrm{Diff}(M)$, but this not the only answer! Another answer is:

$$\Pi_1(M)=\{\gamma: I\to M\}/_{\text{homotops rel. end points}}$$

We will justify why in the following set of propositions/exercises.

Proposition.

• $\Pi_1(M)$ is a groupoid on M:



(i) source/target maps: $s([\gamma])=\gamma(0)$, $t([\gamma])=\gamma(1)$

(ii) multiplication:

$$[\gamma]\cdot[\tau]=\begin{cases}\tau(2t), & 0\le t\le 1/2\\ \gamma(2t-1), & 1/2\le t\le 1\end{cases}$$

(iii) units: $1_x=[x]$, w/ $x(t)=x$, $t\in[0,1]$

(iv) inverse: $[\gamma]^{-1}=[\bar\gamma]$, w/ $\bar\gamma(t)=\gamma(1-t)$, $t\in[0,1]$

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Exercise 1: Show that $\Pi_1(M)$ has a natural structure of a manifold of dimension 2dim for which the underlying topology is the quotient topology on $P(M)/\sim$ induced from the C^0-topology on $P(M)$.

<u>Proposition</u>: $\Pi_1(M)$ is a Lie groupoid integrating $(TM, [\cdot,\cdot], id)$.

<u>Proof</u> (Sketch)

<u>Step 1</u>: Assume M 1-connected. Then

$$\Pi_1(M) \simeq M \times M$$

& statement is easy.

<u>Step 2</u>. If M is not 1-connected, take $p: \tilde M \to M$ the universal covering space so that $M \simeq \tilde M / \pi_1(M)$. Then:

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$$\begin{array}{ccc} \tilde M \times \tilde M & & \\ \downarrow\downarrow & \circlearrowleft \ \pi_1(M) & \\ \tilde M & & \end{array} \quad \Rightarrow \quad \begin{array}{c} \Pi_1(M) \simeq \tilde M \times \tilde M /_{\pi_1(M)} \\ \downarrow\downarrow \\ M \simeq \tilde M /_{\pi_1(M)} \end{array}$$

  - proper & free
  - by groupoid automorphisms

- $\tilde M \times \tilde M$ integrates $(T\tilde M, [\cdot,\cdot], id)$ $\Rightarrow$ $\Pi_1(M)$ integrates $(T\tilde M/_{\pi_1(M)}, [\cdot,\cdot], id)$
$$T\tilde M/_{\pi_1(M)} \simeq TM$$
∎

<u>Poisson Geometry</u>:

It is natural to look for an object $\Sigma(M) := \{\text{ctg paths}\}/_{\text{ctg homotopies}}$

What are cotangent homotopies?

<u>Rmk</u>: In ordinary geometry, it is enough to look at loops. In Poisson geometry this is not true anymore: leaves can be very different!

<u>Ordinary Geometry</u>

- If ∇ is a <u>flat</u> connection on $E \to M$:
$$\gamma_1 \sim \gamma_2 \Rightarrow \tilde\tau_{\gamma_1} = \tilde\tau_{\gamma_2}$$
- If $\alpha \in \Omega^1(M)$ is <u>closed</u>:
$$\gamma_1 \sim \gamma_2 \Rightarrow \int_{\gamma_1} \alpha = \int_{\gamma_2} \alpha$$

<u>Note</u>:
$$\int_\gamma \alpha = \int_0^1 \langle \alpha|_{\gamma(t)}, \dot\gamma(t) \rangle dt$$
$$\int_\gamma dh = h(\gamma(1)) - h(\gamma(0))$$

<u>Poisson Geometry</u>

- If ∇ is a <u>flat</u> contravariant connection on $E \to M$:
$$a_1 \sim a_2 \Rightarrow \tilde\tau_{a_1} = \tilde\tau_{a_2}$$
- If $X \in \mathfrak{X}^1$ is a <u>Poisson v.f.</u>:
$$a_1 \sim a_2 \Rightarrow \int_{a_1} X = \int_{a_2} X$$

<u>Note</u>:
$$\int_a X = \int_0^1 \langle X|_{\gamma(t)}, a(t) \rangle dt$$
$$\int_a X_h = h(\gamma(1)) - h(\gamma(0))$$

Denote by $P_\pi(M)$ the set of cotangent paths of class C^1.

<u>Thm</u> (Crainic & Fernandes, 2002)

Let (M,π) be a Poisson manifold and consider all equivalence relations on $P_\pi(M)$ satisfying the following properties:

(i) //-transport for flat connections is invariant under $\sim$;

(ii) Integration of Poisson v.f. is invariant under $\sim$;

There exists a unique strongest equivalence relation satisfying these properties.

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· <u>Rmk:</u> Actually, the result is only known if we allow non-linear flat connections (analogue of flat Ehresmann connections). I don't know if result is true (even for ordinary geometry!) if we consider only linear connections.

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Explicit description of the strongest equivalence relation (known as cotangent homotopy): Fix ordinary connection ∇ on T^*M

$a_0 \sim a_1$ iff $\exists\, a_\varepsilon \in P_\pi(M)$, $\varepsilon \in [0,1]$, an ordinary homotopy joining a_0 to a_1 for which the unique solution of

$$\begin{cases} \partial_t b = \partial_\varepsilon a + T_\nabla(a,b) \\ b(\varepsilon,0) = 0 \end{cases}$$

satisfies $b(\varepsilon,1) = 0$.

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On $P_\pi(M)$ we take the C^1-topology (i.e., uniform convergence of $a(t)$ & $a'(t)$) and we set:

$$\Sigma(M) := P_\pi(M)/\sim \ \underset{t}{\overset{s}{\rightrightarrows}} M$$

$s([a]) = \gamma(0)$, $t([a]) = \gamma(1)$

$u(x) = [0_x]$, where $0_x \in T^*_x M$

$[a]^{-1} = [\bar{a}]$, where $\bar{a}(t) = a(1-t)$

Given a_1 & $a_2 \in P_\pi(M)$ w/ $s([a_1]) = t([a_2])$:

$$a_1 \cdot a_2(t) := \begin{cases} 2\, a_2(2t), & 0 \le t \le 1/2 \\ 2\, a_1(2t-1), & 1/2 \le t \le 1 \end{cases}$$

At $t = 1/2$ this cotangent path may fail to be C^1.

Exercise 3

Consider a smooth reparametrization $\tilde{\tau} : I \to I$: $\tilde{\tau}(0)=0, \tilde{\tau}(1)=1$ & $\tilde{\tau}'(t) > 0$, $t \in]0,1[$. Given any $a \in P_\pi(M)$ show that $a \sim a^{\tilde{\tau}}$, where $a^{\tau}(t) := \tau'(t)\, a(\tau(t))$.

Hint: Set $a_\varepsilon(t) = ((1-\varepsilon) + \varepsilon\tau'(t))\, a((1-\varepsilon)t + \varepsilon\tau(t))$
$b(\varepsilon,t) = (\tau(t) - t)\, a((1-\varepsilon)t + \varepsilon\tau(t))$

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By the exercise, if we choose τ such that $\tau'(0) = \tau'(1) = 0$, any class $[a]$ has a representative with $a(0) = a(1) = 0$. In fact, we can define:

$$[a_1]\cdot[a_2] := [a_1^{\tau} \cdot a_2^{\tau}]$$

Proposition: With this operation $\Sigma(M)$ is a topological groupoid whose source fibers are 1-connected.

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Proof: All the operations already at the level of $P_\pi(M)$ are continuous.
$\Rightarrow$ All operations on $\Sigma(M)$ are continuous.

- $s^{-1}(x)$ is 1-connected: if $\gamma_\varepsilon = [a_\varepsilon] \in s^{-1}(x)$ is a loop starting at $1_x = [0_x]$, then $h : I \times I \to s^{-1}(x)$, $h(\varepsilon,\tau) := [t \mapsto \tau a_\varepsilon(\tau t)]$ is a homotopy from γ_ε to the trivial loop.

- One still needs to check that s & t are open maps (see Crainic & Fernandes).

∎

There is another description of $\Sigma(M)$, as an infinite dimensional symplectic quotient, which is very useful:

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- $P(T^*M) := \{ a : I \to T^*M \} \supset P_\pi(M)$
- $T_a(P(T^*M)) = \{ X : I \to T(T^*M) \mid X(t) \in T_{a(t)}(T^*M) \}$
- $\omega \in \Omega^2(P(T^*M))$:

$$\omega_a(X, Y) := \int_0^1 \omega_{T^*M}(X(t), Y(t))\, dt$$

RMK: $P(T^*M) \simeq T^* P(M)$ AND $\omega \simeq \omega_{T^* P(M)}$.

Thm (Crainic & Fernandes, Cattaneo & Felder)
Consider the Lie algebra of time-dependent 1-forms vanishing at end-points

$$P_0 \Omega^1(M) = \{ \eta_t \in \Omega^1(M) : \eta_0 = \eta_1 = 0 \}, \quad [\eta_\cdot, \xi_\cdot]_t := [\eta_t, \xi_t]$$

Then this Lie algebra acts on $P(T^*M)$ so that:

(i) The action is tangent to $P_\pi(M)$;

(ii) Orbits in $P_\pi(M)$ have finite codimension $= 2 \dim M$;

(iii) Two cotangent paths $a_0, a_1 \in P_\pi(M)$ are ctg homotopic iff they lie in same orbit.

(iv) Action is hamiltonian with moment map $\mu : P(T^*M) \to [P_0\Omega^1(M)]^*$

$$\langle \mu(a), \eta_\cdot \rangle = \int_0^1 \langle \pi^\#(a(t)) - \frac{d}{dt} p(a(t)), \eta_t \rangle\, dt$$

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Corollary:
Assume that $\Sigma(M)$ is smooth. Then it carries a symplectic form Ω which is multiplicative:

$$\text{graph}(m) \subset \Sigma(M) \times \Sigma(M) \times \overline{\Sigma(M)} \text{ is Lagrangian}$$

$$\Leftrightarrow \quad m^*\Omega = \pi_1^*\Omega + \pi_2^*\Omega \qquad \text{where}$$

$$\begin{tikzcd}
 & \Sigma \times \Sigma \\
 & \Sigma^{(2)} \arrow[u, phantom, "\cup"] \arrow[r, "m"] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \Sigma \\
\Sigma & & \Sigma
\end{tikzcd}$$

The map $s : \Sigma(M) \to M$ (resp. $t : \Sigma(M) \to M$) is Poisson.

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Defn: A Poisson manifold is called integrable if $\Sigma(M)$ is smooth.

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Exercise 4

Let (S,ω) be a symplectic manifold: $\pi = \omega^{-1}$. Show that

$\Sigma(S) = \Pi_1(S)$ and $\Omega = s^*\omega - t^*\omega$

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Exercise 5

Let (M,π) be the zero Poisson structure: $\pi \equiv 0$. Show that

$\Sigma(M) = T^*M$ and $\Omega = \omega_{T^*M}$.

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Example

For a linear Poisson structure $M = \mathfrak{g}^*$, one can show that

$\Sigma(\mathfrak{g}^*) = T^*G \rightrightarrows \mathfrak{g}^*$, where G is the 1-connected Lie group integrating $\mathfrak{g}$, $\Omega = \omega_{T^*G}$, and the structure maps are:

$$s(\alpha_g) = (d_e L_g)^*\alpha_g, \quad t(\alpha_g) = (d_e R_g)^*\alpha_g$$

$$\alpha_g \cdot \beta = \gamma_{gh} \quad \text{where} \quad \left| \begin{array}{l} (d_h L_g)^*\gamma_{gh} = \beta_h \\ (d_g R_h)^*\gamma_{gh} = \alpha_g \end{array} \right.$$

Exercise 6

Show that $\Sigma(\mathfrak{g}^*) \simeq G \ltimes \mathfrak{g}^*$, where G acts on $\mathfrak{g}^*$ via the coadjoint action. What is the expression of Ω under this isomorphism?

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As one can guess from the previous examples/exercises there can be other integrations of a Poisson manifold (M,π):

- $S \times \bar{S} \rightrightarrows S$ & $\Pi_1(S) \rightrightarrows S$ both integrate (S,ω)
- $T^*G \rightrightarrows \mathfrak{g}^*$ & $T^*G' \rightrightarrows \mathfrak{g}^*$ w/ $Lie(G) = Lie(G') = \mathfrak{g}$ both integrate $\mathfrak{g}^*$

__Thm__ (Weinstein)

Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a Lie groupoid with a multiplicative symplectic form Ω. Then there exists a unique Poisson structure on M such that $s: (\mathcal{G},\Omega) \to (M,\pi)$ (resp. $t: (\mathcal{G},\Omega) \to (M,\pi)$) is a Poisson map (resp. anti-Poisson). If $\mathcal{G}$ is source 1-connected then $(\mathcal{G}, \Omega_{\mathcal{G}}) \simeq (\Sigma(M), \Omega)$.

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__Proof__ __(Sketch)__

- Given $(\mathcal{G},\Omega)$ one constructs a Poisson bracket on M observing that multiplicativity of Ω implies that $(s\text{-fibers})^{\perp_\Omega} = (t\text{-fibers})$, so that the Poisson bracket of functions constant on s-fibers is a function constant on s-fibers.

- Given $(\mathcal{G},\Omega)$ source 1-connected one constructs a map $\Phi: \mathcal{G} \to \Sigma(M)$ as follows:

$$\mathcal{G} \ni g \longmapsto \left[a(t) = \frac{d}{ds} g(s) g(t)^{-1} \Big|_{s=t} \right] \in P_\pi(M)/\sim$$

where $g(t)$ is any path w/ $\left| \begin{array}{l} g(1) = g \\ g(0) = 1_{s(g)} \\ s(g(t)) = s(g) \end{array} \right.$

And we use Ω to identify $\operatorname{Ker} d_M s \simeq T^*M$.

__Exercise 7__: Show that this map is a groupoid isomorphism.

$\boxtimes$

Thm (Lie I for Poisson str.)

Any Poisson structure π which is induced by some symplectic groupoid $(\mathcal{G}, \Omega) \rightrightarrows M$ is integrable

Proof (Sketch)

We need to show that $\Sigma(M)$ is smooth.

Step 1: Show that we can assume that $\mathcal{G}$ is source-connected

(Hint: Consider the connected component of the identity section of $(\mathcal{G}, \Omega)$)

Step 2: Show that we can assume that $\mathcal{G}$ is source 1-connected.

(Hint: Let $\tilde{\mathcal{G}} = \{\text{paths in } s\text{-fibers starting at identity}\} / \text{homotopy in } s\text{-fibers}$ and verify that $\Phi: \tilde{\mathcal{G}} \to \mathcal{G}$, $[\gamma] \mapsto \gamma(1)$, is an étale morphism of topological groupoids.)

Step 3: Apply the previous proposition to conclude that $(\mathcal{G}, \Omega) \simeq (\Sigma(M), \Omega)$ so $\Sigma(M)$ is smooth.

$\boxtimes$

We now turn to the integration of Poisson maps, for which we need another fundamental concept:

Defn: A submanifold N of a Poisson manifold (M, π) is called coisotropic if $\pi^{\#}(TN)^{\circ} \subset TN$.

—/—

Examples

1. Every codimension 1 submanifold is coisotropic
2. A subspace $W \subset \mathfrak{g}$ is a Lie subalgebra iff $W^{\circ} \subset \mathfrak{g}^*$ is coisotropic
3. Every Poisson submanifold is coisotropic.
4. A map $\phi: (M_1, \pi_1) \to (M_2, \pi_2)$ is Poisson iff $\operatorname{graph} \phi \subset (M_1 \times \bar{M}_2, \pi_1 \times (-\pi_2))$ is a coisotropic submanifold

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Theorem (Cattaneo & Xu)

Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a symplectic groupoid and let $(\mathcal{H} \rightrightarrows N) \subset (\mathcal{G} \rightrightarrows M)$ be a Lagrangian subgroupoid. Then N is a coisotropic submanifold of (M, π).

Conversely, if N is a coisotropic submanifold of an integrable Poisson manifold (M, π) then:

$$\mathcal{H} = \{ a \in P_\pi(M) \mid a(t) \in (TN)^\circ \} / \sim \ \subset \Sigma(N)$$

is a Lagrangian subgroupoid.

— / —

Corollary (Lie II for Poisson manifolds)

Let $\phi : (M_1, \pi_1) \to (M_2, \pi_2)$ be a Poisson map between integrable Poisson manifolds. Then there exists a Lagrangian subgroupoid $(\mathcal{H} \rightrightarrows \mathrm{graph}(\phi)) \subset \Sigma(M_1) \times \overline{\Sigma(M_2)} \rightrightarrows M_1 \times \overline{M}_2$ integrating ϕ.

— / —

This result can be extended to a correspondence:

"Poisson relations" $\longleftrightarrow$ "symplectic relations"

We have not mention Lie III. In fact, there are examples of non-integrable Poisson manifolds, and there is an "obstruction theory" for integrability. We will not go into this, but we point out the following important relation w/ the concept of a "proper" Poisson map

Defn:

A Poisson map $\phi : (M_1, \pi_1) \to (M_2, \pi_2)$ is said to be <u>complete</u> if:

$$X_h \in \mathfrak{X}(M_2) \text{ complete v.f.} \Rightarrow X_{h\circ\phi} \in \mathfrak{X}(M_1) \text{ complete v.f.}$$

— / —

Exercise 8

Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be any symplectic groupoid. Show that the source map $s : (\mathcal{G}, \Omega) \to (M, \pi)$ is a complete Poisson map.

— / —

Thm (Crainic & Fernandes)

A Poisson manifold (M, π) is integrable iff it admits a complete symplectic realization $\phi : (S, \omega) \to (M, \pi)$.

(surjective + submersion + Poisson)

— / —

Rmk: One aspect we have "hidden under the rug" is that one is often forced to consider <u>non-Hausdorff</u> Lie groupoids.

Exercise 9: On $M = \mathbb{R}^3 \setminus \{0\}$ consider the Poisson structure:

$$\pi = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$

Show that $\Sigma(M)$ is a smooth, non-Hausdorff, Lie groupoid.

— / —